LECTURES ON RIEMANN SURFACES

BY

R. C. GUNNING

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Preface.

These are notes for a course of lectures given at Princeton University during the academic year 1965-66. The subject of the lectures was compact Riemann surfaces, considered as complex analytic manifolds. There are already several expositions of this subject from basically the same point of view; the foremost is undoubtedly Hermann Weyl's classic "Die Idee der Riemannschen Fläche," and most of the later treatments have followed Weyl's approach to a large degree. During recent years there has been considerable activity in the study of complex analytic manifolds of several dimensions, and various new tools and approaches have been developed. The aim of the lectures, in addition to treating of a beautiful subject for its own sake, was to introduce the students to some of these techniques in the case of one complex variable, where things are simpler and the results more complete.

The material covered is indicated by the table of contents. No familiarity with manifolds, sheaves, or sheaf cohomology was assumed, so those subjects are developed ab initio, although only so far as necessary for the purposes of the lectures. On the other hand, no attempt was made to discuss in detail the topology of surfaces; for that is really another subject, and there are very good treatments available elsewhere. The basic analytic tool used was the Serre duality theorem, rather than the theory of harmonic integrals or harmonic functions. The detailed treatment of the analytic properties of compact Riemann surfaces begins only in §7. Unfortunately, there was not enough time to get very far in the discussion; so the lectures have the air of being but an introduction to the subject. This may explain some of the surprising omissions, also. I hope to have an opportunity to continue the discussion further sometime.

With the possible exception of parts of §9, there is nothing really new here. References to the literature are scattered throughout, with no attempt at completeness. In addition to these and to the book of Hermann Weyl, the following general references should be mentioned here: Paul Appell and Edouard Goursat, "Théorie des Functions Algébriques," (Gauthier-Villars, 1930); Kurt Hensel and Georg Landsberg, "Theorie der algebraischen Funktionen einer Variablen," (Teubner, 1902; Chelsea, 1962); and Jean-Pierre Serre, "Corps locaux," (Hermann, 1962).

I should like to express my thanks here to Richard Hamilton, Henry Laufer, and Richard Mandelbaum for many suggestions and improvements; and to Elizabeth Epstein for typing the manuscript.

Princeton, New Jersey R. C. Gunning

May, 1966

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SOME FREQUENTLY USED SYMBOLS

$\mathbb{C}$	complex field
$\mathcal{C}$	sheaf of germs of continuous functions, 22
$\mathcal{D}$	divisor, 49
$\mathcal{E}$	sheaf of germs of C^{∞} differential forms, 68
$\mathcal{K}$	sheaf of germs of distributions, 83
$\mathcal{O}$	sheaf of germs of holomorphic functions, 20
$\mathcal{O}^{1,0}$	sheaf of germs of Abelian differentials, 72
$\mathbb{P}$	projective line
κ	canonical line bundle, 78
$\gamma(\xi) =$	$\dim \Gamma(M, \mathcal{O}(\xi))$, 111

§1. Fundamental definitions.

(a) The field of real numbers will be denoted by $\mathbb{R}$, and the field of complex numbers by $\mathbb{C}$; both are topological fields, with the familiar structures. The cartesian product of $\mathbb{R}$ with itself n times, the usual euclidean n-space, will be denoted by $\mathbb{R}^n$; note that, as topological spaces, $\mathbb{C}$ and $\mathbb{R}^2$ can be identified with one another.

Definition. An n-dimensional topological manifold is a Hausdorff space M such that every point $p \in M$ has an open neighborhood homeomorphic to an open cell in $\mathbb{R}^n$.

Let M be an n-dimensional topological manifold. A coordinate covering $\{U_\alpha, z_\alpha\}$ of M consists of an open covering $\{U_\alpha\}$ of M together with homeomorphisms

$$z_\alpha : U_\alpha \longrightarrow V_\alpha$$

from the subsets $U_\alpha \subset M$ to open cells $V_\alpha \subset \mathbb{R}^n$. The sets U_α will be called coordinate neighborhoods, and the mappings z_α will be called coordinate mappings. By definition, any topological manifold admits a coordinate covering. On each non-empty intersection $U_\alpha \cap U_\beta$ two different homeomorphisms into $\mathbb{R}^n$ are defined; the compositions

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \longrightarrow z_\alpha(U_\alpha \cap U_\beta)$$

will be called the coordinate transition functions of the coordinate covering. Thus for a point $p \in U_\alpha \cap U_\beta$, the two coordinate mappings are related by $z_\alpha(p) = f_{\alpha\beta}(z_\beta(p))$. The following diagram should illustrate these concepts.

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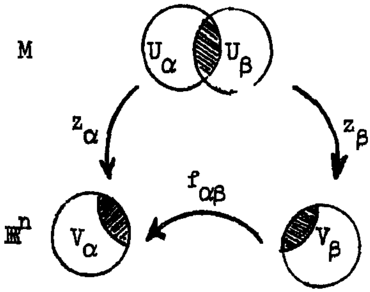


($U_\alpha \cap U_\beta$ shaded)

($z_\alpha(U_\alpha \cap U_\beta)$ and $z_\beta(U_\alpha \cap U_\beta)$ shaded)

Note that a description of the sets $\{V_\alpha\}$ and of the mappings $\{f_{\alpha\beta}\}$ is enough to reconstruct the original manifold M ; for M can be obtained from the disjoint union of all the sets $\{V_\alpha\}$ by identifying a point $z_\alpha \in V_\alpha$ and a point $z_\beta \in V_\beta$ whenever $z_\alpha = f_{\alpha\beta}(z_\beta)$. As a convenient abbreviation, the sets V_α will sometimes also be called coordinate neighborhoods for M .

Suppose that $\{U_\alpha, z_\alpha\}$ and $\{U'_\alpha, z'_\alpha\}$ are two coordinate coverings of the manifold M . The <u>union</u> $\{U_\alpha, z_\alpha\} \cup \{U'_\alpha, z'_\alpha\}$ of these two coordinate coverings is the new coordinate covering consisting of all the coordinate neighborhoods and mappings from the two given coverings. It is important to observe that the set of coordinate transition functions for the union $\{U_\alpha, z_\alpha\} \cup \{U'_\alpha, z'_\alpha\}$ is properly larger than the union of the sets of coordinate transition functions for $\{U_\alpha, z_\alpha\}$ and for $\{U'_\alpha, z'_\alpha\}$; for, in addition to the coordinate transition functions associated to the intersections $U_\alpha \cap U_\beta \neq \emptyset$ and those associated to the intersections $U'_\alpha \cap U'_\beta$, there are the coordinate transition functions associated to all the intersections $U_\alpha \cap U'_\beta \neq \emptyset$.

For most of the subsequent discussion, the manifolds under discussion will be of dimension 2; and the coordinate neighborhoods

$\{V_\alpha\}$ will be considered as lying in the complex line $\mathbb{C}$ rather than in $\mathbb{R}^2$. The coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$ are hence continuous complex-valued functions defined on subsets of the complex line $\mathbb{C}$. All manifolds will be assumed connected.

(b) Coordinate coverings having particular properties can be used to impose a great many additional structures on topological manifolds. A coordinate covering $\{U_\alpha, z_\alpha\}$ of a 2-dimensional manifold M will be called a <u>complex analytic coordinate covering</u> if all the coordinate transition functions are holomorphic (that is, complex analytic) functions. Two complex analytic coordinate coverings will be called <u>equivalent</u> if their union is also a complex analytic coordinate covering. It is easy to see that this is indeed a proper equivalence relation. (Since symmetry and reflexivity are trivial, it is only necessary to verify transitivity. Consider, therefore, complex analytic coordinate coverings $\{U_\alpha, z_\alpha\}$ equivalent to $\{U'_\alpha, z'_\alpha\}$, and $\{U'_\alpha, z'_\alpha\}$ equivalent to $\{U''_\alpha, z''_\alpha\}$. For any point $p \in U_\alpha \cap U''_\beta$ there will be a coordinate neighborhood U'_γ such that $p \in U'_\gamma$; and in a suitably small open neighborhood of $z''_\beta(p)$ it is obvious that $f_{\alpha\beta} = f_{\alpha\gamma} \circ f_{\gamma\beta}$, with the obvious notation. Since $f_{\alpha\gamma} = z_\alpha \circ (z'_\gamma)^{-1}$ and $f_{\gamma\beta} = z'_\gamma \circ (z''_\beta)^{-1}$ are holomorphic by hypothesis, and since any composition of holomorphic functions is again holomorphic, it follows that $f_{\alpha\beta} = z_\alpha \circ (z''_\beta)^{-1}$ is holomorphic near $z''_\beta(p)$. This holds for all points $p \in U_\alpha \cap U''_\beta$, and that suffices to prove the desired transitivity.) An equivalence class of complex analytic coordinate coverings of M will be called a complex analytic structure (or simply a complex structure) on M. In the traditional terminology, a surface M with a fixed complex structure is called a Riemann surface.

It should be noted that the only property of holomorphic functions needed for the preceding definitions is that holomorphic functions are closed under composition, whenever composition is defined; this will be called the pseudogroup property. Thus, for any class of homeomorphisms with the pseudogroup property, it is possible to introduce a corresponding structure on manifolds. (For a general discussion of pseudogroups and their classification, see for instance V. W. Guillemin and S. Sternberg, "An algebraic model of transitive differential geometry," Bull. Amer. Math. Soc. 70 (1964), 16-47, and the literature mentioned there.) As an example, the set of homeomorphisms of $\mathbb{R}^2$ which possess continuous partial derivatives of all orders has the pseudogroup property; the corresponding structure on a manifold will be called a differentiable structure, (or more precisely, a C^∞ differentiable structure). Since holomorphic functions are also infinitely differentiable, a complex analytic structure on a manifold belongs to a unique differentiable structure; the complex analytic structure will be said to be subordinate to the differentiable structure. These differentiable structures do not play a significant role in the study of Riemann surfaces, since in fact there is a unique differentiable structure on any 2-dimensional manifold; (sée J. R. Munkres, "Obstructions to the smoothing of piecewise-differentiable homeomorphisms," Annals of Math. 72 (1960), 521-554). This situation is quite different in the higher-dimensional cases, however.

Some further pseudogroups and related structures, subordinate to complex analytic structures, will appear later in this discussion.

(c) Let M be a Riemann surface, and $\{U_\alpha, z_\alpha\}$ be a complex analytic coordinate covering belonging to the given complex analytic structure on M. A mapping f from an open subset $U \subset M$ into the complex line $\mathbb{C}$ will be called a <u>holomorphic function</u> in U if for each intersection $U \cap U_\alpha \neq \emptyset$ the mapping

$$f \circ z_\alpha^{-1}: z_\alpha(U \cap U_\alpha) \longrightarrow \mathbb{C}$$

is a holomorphic function in the subset $z_\alpha(U \cap U_\alpha) \subset \mathbb{C}$. It is easy to see that the property that a function be holomorphic is independent of the choice of complex analytic coordinate covering belonging to the complex structure; the verification will be left to the reader. The set of all functions holomorphic in U will be called the <u>ring of holomorphic functions in</u> U, and will be denoted by $\mathcal{O}_U$; this set is clearly a ring, under the pointwise addition and multiplication of functions, and contains the constant-valued functions as a subring isomorphic to $\mathbb{C}$. In terms of the differentiable structure associated to the complex analytic structure of M, a <u>differentiable function</u>, (or more precisely, a C^∞ complex-valued function) is defined correspondingly as a mapping $f: U \longrightarrow \mathbb{C}$ such that for $f \circ z_\alpha^{-1}$ is C^∞ in each set $z_\alpha(U \cap U_\alpha) \neq \emptyset$; the ring of differentiable functions in U will be denoted by $\mathcal{E}_U^\infty$, and the ring of all continuous complex-valued functions in U will be denoted by $\mathcal{E}_U$. Note that these

these rings are related as follows: $\mathbb{C}_U \subset \mathcal{O}_U \subset \mathcal{C}^{\infty}_U \subset \mathcal{C}_U$.

The <u>field of meromorphic functions</u> in an open subset $U \subset M$ can be defined correspondingly as well, and will be denoted by $\mathcal{M}_U$. It should be noted that a meromorphic function is not, properly speaking, a mapping into $\mathbb{C}$; thus the field $\mathcal{M}_U$ is not really comparable to the rings $\mathcal{C}^{\infty}_U$ or $\mathcal{C}_U$. An interpretation of $\mathcal{M}_U$ as a set of mappings will be given later, however; (see part (e) of §1).

Let $f \in \mathcal{O}_U$ for an open set $U \subset M$, and consider a point $p \in U$. The order of <u>the function</u> f <u>at the point</u> p is defined to be the order of the holomorphic function $f \circ z_{\alpha}^{-1}$ at the point $z_{\alpha}(p) \in \mathbb{C}$, for any coordinate neighborhood U_{α} containing p; and this order will be denoted $\nu_p(f)$. Recall that the order of a holomorphic function of a complex variable z at a point $z = a$ is the order of the first non-zero coefficient in the Taylor expansion of the function in terms of the variable $z - a$; and note that the order is independent of the complex analytic coordinate covering belonging to the complex analytic structure of M. Of course, $\nu_p(f) \geqq 0$ at all points $p \in U$; and $\nu_p(f) > 0$ only at a discrete subset of points of U, if f is not identically zero. Similarly, for a meromorphic function $f \in \mathcal{M}_U$, the order can be defined, and will also be denoted $\nu_p(f)$; in this case, $\nu_p(f)$ can be negative as well, but again $\nu_p(f) \neq \emptyset$ except at a discrete subset of points of U.

<u>Lemma 1.</u> If M is a compact connected Riemann surface, then $\mathcal{O}_M = \mathbb{C}$.

Proof. As noted above, the ring of constant-valued functions is a subring $\mathbf{C} \subset \mathcal{O}_M$. If $f \in \mathcal{O}_M$, then since M is compact, the function $|f|$ must attain its maximum at some point $p \in M$. If U_α is a coordinate neighborhood containing p, then $|f \circ z_\alpha^{-1}|$ reaches its maximum at an interior point $z_\alpha(p) \in z_\alpha(U \cap U_\alpha)$; hence $f \circ z_\alpha^{-1}$ must be constant in an open neighborhood of $z_\alpha(p)$, and f must be constant in an open neighborhood of p. It follows directly from the identity theorem in function theory that the interior of the set of points at which f is constant is both open and closed in M; since that set is non-empty, and M is connected, f is actually constant on M and thus $\mathcal{O}_M \subset \mathbf{C}$.

(d) The notion of a holomorphic function on a Riemann surface can be generalized as follows. Let M and M' be two Riemann surfaces; and let $\{U_\alpha, z_\alpha\}$ and $\{U'_\alpha, z'_\alpha\}$ be complex analytic coordinate coverings belonging to the two given complex structures. A mapping $f: M \longrightarrow M'$ is called a <u>holomorphic mapping</u> if, for any point $p \in M$ and for any coordinate neighborhoods $U_\alpha \subset M$, $U'_\beta \subset M'$ such that $p \in U_\alpha$ and $f(p) \in U'_\beta$, the function $z'_\beta \circ f \circ z_\alpha^{-1}$ is a holomorphic function in the usual sense in some open neighborhood of the point $z_\alpha(p) \in \mathbf{C}$. It is easy to see that the property that a mapping be holomorphic is independent of the choices of complex analytic coordinate coverings belonging to the two complex analytic structures; the verification will be left to the reader. Note that a holomorphic mapping is necessarily continuous; morever, such a mapping is also differentiable (or more precisely, C^∞) in terms of the differentiable structures on the surfaces. A holomorphic function is the special case of a holomorphic mapping from M to the Riemann surface $\mathbf{C}$.

These holomorphic mappings can be characterized very conveniently by their effects on holomorphic functions. Let $f: M \longrightarrow M'$ be any continuous mapping between two Riemann surfaces M, M', and let $U' \subset M'$ be any open subset of M'. The map f induces a homomorphism $f^*: \mathcal{C}_{U'} \longrightarrow \mathcal{C}_U$, where $U = f^{-1}(U') \subset M$, by defining $f^*(h_{U'}) = h_{U'} \circ f$; in particular, $f^*(\mathcal{O}_{U'}) \subset \mathcal{C}_U$ is a well defined subring.

<u>Lemma 2.</u> A continuous mapping $f: M \longrightarrow M'$ between two Riemann surfaces is a holomorphic mapping if and only if $f^*(\mathcal{O}_{U'}) \subset \mathcal{O}_U$ for every open subset $U' \subset M'$, where $U = f^{-1}(U')$.

Proof. Select any point $p \in M$, and coordinate neighborhoods $U_\alpha \subset M$, $U'_\beta \subset M'$ such that $p \in U_\alpha$ and $f(p) \in U'_\beta$. If the mapping f satisfies the conditions of the lemma, then considering in particular the holomorphic function $z'_\beta \in \mathcal{O}_{U'_\beta}$, it follows that $f^*(z'_\beta) = z'_\beta \circ f$ is holomorphic, that is, that $z'_\beta \circ f \circ z_\alpha^{-1}$ is holomorphic; therefore f is a holomorphic mapping. Conversely, if the mapping f is holomorphic and if h' is a holomorphic function in an open neighborhood $U' \subset U'_\beta$ of $f(p)$, then $f^*(h') \circ z_\alpha^{-1} = h' \circ f \circ z_\alpha^{-1} = (h' \circ (z'_\beta)^{-1}) \circ (z'_\beta \circ f \circ z_\alpha^{-1})$ is a holomorphic function in a neighborhood of $z_\alpha(p)$, since it is the composition of the holomorphic functions $h' \circ (z'_\beta)^{-1}$ and $z'_\beta \circ f \circ z_\alpha^{-1}$; hence f satisfies the conditions of the lemma.

Extending the previous discussion, a topological homeomorphism $f: M \longrightarrow M'$ between two Riemann surfaces will be called a <u>holomorphic</u>

isomorphism if both mappings f and f^{-1} are holomorphic mappings; the Riemann surfaces M and M' will be called isomorphic if there is a holomorphic isomorphism between them. Clearly, the real interest lies in the isomorphism classes of Riemann surfaces.

(e) The simplest example of a Riemann surface is the complex line $\mathbb{C}$; of course, since any subdomain of a Riemann surface is again a Riemann surface, subdomains of $\mathbb{C}$ are also contenders for the title of simplest Riemann surface. As for compact manifolds, the 2-sphere is clearly the simplest case; thus there arises the question whether the 2-sphere admits a complex structure.

The 2-sphere, considered merely as a topological manifold M, can be given a coordinate covering as follows. Let $n, s \in M$ be two distinct points of M, which can be envisaged as the north and south poles of the 2-sphere. The open sets $U_0 = M-s$ and $U_1 = M-n$ cover M, and are topological cells; so select some homeomorphisms z_α between these sets and the standard 2-cell, which can be taken to be the full euclidean plane $\mathbb{C}$. This describes a coordinate covering $\{U_0, z_0\}$, $\{U_1, z_1\}$ of M. There is no loss of generality in supposing further that $z_0(n) = 0 \in \mathbb{C}$, and $z_1(s) = 0 \in \mathbb{C}$; so the coordinate transition function f_{01} is a homeomorphism

$$f_{01}: (\mathbb{C} - 0) \longrightarrow (\mathbb{C} - 0),$$

which takes the interior of an open topological disc about the origin in $\mathbb{C} - 0$ onto the exterior of another such set. Conversely of course, any choice of such a homeomorphism f_{01} can be realized as the coordinate transition function for a coordinate covering of the 2-sphere M of the above form; the question of the existence of a complex analytic structure on the 2-sphere thus becomes merely the question of the existence of a complex analytic homeomorphism $z_0 = f_{01}(z_1)$ of this form. In particular, the function $z_0 = f_{01}(z_1) = 1/z_1$ will serve the purpose; and the 2-sphere with this complex structure will be called the <u>complex projective line</u>, and denoted by $\mathbb{P}$.

It should be noted that the complex line $\mathbb{P}$ as described above is actually the one-dimensional complex projective space in the usual sense; (see for instance W. V. D. Hodge and D. Pedoe, Methods of Algebraic Geometry, volume I, chapter V, (Cambridge University Press, 1953)). Thus let $\mathbb{C}^* = \mathbb{C} - 0$, considered as a multiplicative group; and let $\mathbb{C}^*$ act as a transformation group on the space $\mathbb{C}^2 - \underline{0} = \{(\zeta_1,\zeta_2) \neq (0,0)\}$ by $t\cdot(\zeta_0,\zeta_1) = (t\zeta_0, t\zeta_1)$ for $t \in \mathbb{C}^*$. The quotient space $(\mathbb{C}^2 - \underline{0})/\mathbb{C}^*$ is the one-dimensional complex projective space; each point of the projective space can be represented by a pair of complex numbers $(\zeta_1,\zeta_2) \neq (0,0)$ called the <u>homogeneous coordinates</u> of the point, this representation of course being far from unique. The two sets

$$U_0 = \{(\zeta_0,\zeta_1) \in \mathbb{C}^2 | \zeta_0 \neq 0\}/\mathbb{C}^*, \qquad U_1 = \{(\zeta_0,\zeta_1) \in \mathbb{C}^2 | \zeta_1 \neq 0\}$$

cover the projective space, and each can be mapped in a one-one manner onto the complex line $\mathbb{C}$ by a mapping

$$z_0 = \zeta_1/\zeta_0 \quad \text{or} \quad z_1 = \zeta_0/\zeta_1 \, .$$

Then $\{U_0, z_0\}$ and $\{U_1, z_1\}$ form a coordinate covering of the space, exhibiting it as the Riemann surface $\mathbb{P}$ described above.

One trivial property of the complex projective line deserves note here. Consider an arbitrary Riemann surface M, and a holomorphic mapping $f: M \longrightarrow \mathbb{P}$. In an open neighborhood U of each point $p \in M$ the mapping f can be described in terms of the homogeneous coordinates on $\mathbb{P}$ by $f(p) = (f_0(p), f_1(p))$, where $f_\alpha(p)$ are holomorphic functions on U in the usual sense. Then the quotient $f_0(p)/f_1(p)$ is a meromorphic function in U; this function is clearly independent of the choice of homogeneous representation, and so is defined throughout the Riemann surface M. Conversely, any meromorphic function on M can be represented locally as the quotient of holomorphic functions, $f_0(p)/f_1(p)$; and then $f(p) = (f_0(p), f_1(p))$ is a well-defined holomorphic mapping $f: M \longrightarrow \mathbb{P}$. That is to say, the meromorphic functions on M are in natural one-to-one correspondence with the holomorphic mappings $f: M \longrightarrow \mathbb{P}$.

(f) The next simplest compact 2-manifold is a surface of genus 1, a torus; complex analytic structures are also easy to describe in this case. In the complex line $\mathbb{C}$ select any two complex numbers ω_1, ω_2 which are linearly independent over the reals; so ω_i are non-zero complex numbers, and $\omega_1/\omega_2 \notin \mathbb{R}$. The numbers ω_1, ω_2 generate a subgroup $\Lambda \subset \mathbb{C}$, namely

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 | n_1 n_2 \in \mathbb{Z} = \text{additive group of integers}\}.$$

The quotient group $\mathbb{C}/\Lambda$ is a well-defined topological space, a surface of genus 1. It is evident that $\mathbb{C}/\Lambda$ has a natural structure inherited from that of $\mathbb{C}$; as coordinate neighborhoods in $\mathbb{C}/\Lambda$ take open subsets of $\mathbb{C}$ which contain no points congruent to one another modulo Λ.

In discussing the sphere, a single complex analytic structure was described; indeed, it will later be shown that there is a unique complex structure on the sphere. In discussing the torus, there were two arbitrary parameters involved in the description of the complex analytic structure, the two constants ω_1, ω_2. It is natural to ask whether there are different complex analytic structures on the torus, corresponding to various choices of the parameters ω_1, ω_2; this is indeed so, showing thusly that a given topological surface may carry a variety of inequivalent complex analytic structures. Suppose then that $\Lambda = \{n_1\omega_1 + n_2\omega_2 | n_i \in \mathbb{Z}\}$ and $\Lambda' = \{n_1\omega_1' + n_2\omega_2' | n_i \in \mathbb{Z}\}$ are two lattice subgroups of $\mathbb{C}$, with associated Riemann surfaces $M = \mathbb{C}/\Lambda$ and $M' = \mathbb{C}/\Lambda'$. If the surfaces M and M' are isomorphic, there is a topological homeomorphism $f: M \longrightarrow M'$ such that f and f^{-1} are holomorphic mappings. Now the mapping f lifts to a mapping F from the universal covering surface $\mathbb{C}$ of $M = \mathbb{C}/\Lambda$ to the universal covering surface $\mathbb{C}$ of $M' = \mathbb{C}/\Lambda'$; and in view of the definition of the complex structures on M and M', the mapping $F: \mathbb{C} \longrightarrow \mathbb{C}$ must be a complex analytic mapping. Moreover, since F arises from the homeomorphism $f: \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda'$, it follows that

$$(1) \qquad \begin{cases} F(z + \omega_1) = F(z) + a_{11}\omega_1' + a_{12}\omega_2', \\ F(z + \omega_2) = F(z) + a_{21}\omega_1' + a_{22}\omega_2', \end{cases}$$

for some integers $a_{ij} \in \mathbb{Z}$, such that $a_{11}a_{22} - a_{12}a_{21} = \pm 1$. Differentiating the above equations, $F'(z+\omega_1) = F'(z)$ and $F'(z+\omega_2) = F'(z)$; so $F'(z)$ is invariant under Λ, hence determines a holomorphic function on $M = \mathbb{C}/\Lambda$. Since $\mathbb{C}/\Lambda$ is compact, it follows from Lemma 1 that the function $F'(z)$ must be constant on M, hence $F'(z) \equiv c$; and therefore $F(z) \equiv cz + d$ for some constants c, d. There is of course no loss of generality in translating the surfaces so that $d = 0$; so $F(z) \equiv cz$, for some complex constant $c \neq 0$. Now from equations (1) it follows that

$$(2) \qquad c\omega_1 = a_{11}\omega_1' + a_{12}\omega_2', \qquad c\omega_2 = a_{21}\omega_1' + a_{22}\omega_2' .$$

The complex numbers $\omega = \omega_1/\omega_2$ and $\omega' = \omega_1'/\omega_2'$ associated to the complex analytic structures of M and M' are therefore related by

$$(3) \qquad \omega = \frac{a_{11}\omega' + a_{12}}{a_{21}\omega' + a_{22}},$$

for some integers a_{ij} such that $a_{11}a_{22} - a_{12}a_{21} = \pm 1$. Conversely, if ω and ω' are so related, there is a complex constant $c \neq 0$ such that (2) holds, and the function $F(z) \equiv cz$ then satisfies (1); and therefore M and M' are isomorphic. Consequently, the Riemann surfaces $M = \mathbb{C}/\Lambda$ and $M' = \mathbb{C}/\Lambda'$ are isomorphic if and only if $\omega = \omega_1/\omega_2$ and $\omega' = \omega_1'/\omega_2'$ are related as above. The set of all possible complex structures of the above form in a complex torus thus correspond to all non-real complex numbers ω modulo the equivalence relation (3); for a more detailed description of the latter relation, see for instance J. Lehner, Discontinuous Groups and Automorphic Functions, Chapter XI, (American Mathematical Society, 1964).

§2. Sheaves

(a) Sheaves have proved to be a very useful tool in the theory of functions of several complex variables, and have occasionally been used in one complex variable as well. For the purposes of an eventual simplicity and of convenience of generalization, they will be used systematically throughout the present discussion of Riemann surfaces. However, no previous acquaintance with sheaves will be assumed here. Those readers already familiar with the general properties of sheaves, and with the cohomology theory of sheaves, can readily skip this and the following section.

Definition.[1] A sheaf (of abelian groups) over a topological space M is a topological space $\mathcal{S}$, together with a mapping $\pi: \mathcal{S} \longrightarrow M$, such that:

(i) π is a local homeomorphism;

(ii) for each point $p \in M$ the set $\mathcal{S}_p = \pi^{-1}(p) \subset \mathcal{S}$ has the structure of an abelian group;

(iii) the group operations are continuous in the topology of $\mathcal{S}$.

The third condition in the above definition is, more explicitly, the following. In the cartesian product $\mathcal{S} \times \mathcal{S}$ with the product topology introduce the subset

$$\mathcal{S} \circ \mathcal{S} = \{(s_1, s_2) \in \mathcal{S} \times \mathcal{S} \mid \pi(s_1) = \pi(s_2)\},$$

[1] Throughout this and the following section, the discussion will be limited to sheaves of abelian groups; it is left to the reader to note the obvious modifications necessary to the consideration of sheaves of rings or fields, etc.

with the topology it inherits from the imbedding $\mathcal{S} \circ \mathcal{S} \subset \mathcal{S} \times \mathcal{S}$. The mapping $\mathcal{S} \circ \mathcal{S} \longrightarrow \mathcal{S}$ given by $(s_1, s_2) \in \mathcal{S} \circ \mathcal{S} \longrightarrow s_1 - s_2 \in \mathcal{S}$ is well-defined; the condition is that it be continuous. In the sheaf $\mathcal{S}$, the mapping $\pi: \mathcal{S} \longrightarrow M$ is called the projection; and the set $\mathcal{S}_p = \pi^{-1}(p)$ is called the stalk over p . Each stalk is an abelian group, although different stalks may be quite different groups.

As a simple example, let G be any abelian group with the discrete topology; let $\mathcal{S}$ be the space $\mathcal{S} = G \times M$ with the product topology, and $\pi: G \times M \longrightarrow M$ be the natural projection mapping. Then $\mathcal{S}$ is a sheaf over M , called a constant sheaf.

Let $\pi: \mathcal{S} \longrightarrow M$ be a sheaf, and $U \subset M$ be an open subset of M . A section of the sheaf $\mathcal{S}$ over U is a continuous mapping $f: U \longrightarrow \mathcal{S}$ such that $\pi \circ f: U \longrightarrow U$ is the identity mapping; note that necessarily $f(p) \in \mathcal{S}_p = \pi^{-1}(p)$ for any $p \in U$. The set of all sections of $\mathcal{S}$ over U will be denoted by $\Gamma(U, \mathcal{S})$. For any point $s \in \mathcal{S}$ there must be an open neighborhood V of s in $\mathcal{S}$ such that

$$\pi|V: V \longrightarrow U$$

is a homeomorphism between V and an open subset $U \subset M$; the inverse map $(\pi|V)^{-1}: U \longrightarrow V$ is also a homeomorphism, hence is indeed a section of $\mathcal{S}$ over U . Therefore each point $s \in \mathcal{S}$ is contained in the image of some section; and the images of all such sections form a basis for the open neighborhoods of s . As a consequence of this, if $f, g \in \Gamma(U, \mathcal{S})$ and if $f(p_o) = g(p_o)$ for some point $p_o \in U$, then $f(p) = g(p)$ for all points p of some open set U' with

$p_o \in U' \subset U$. Now, if $f, g \in \Gamma(U, \mathcal{S})$ again, the mapping

$$f \times g: p \in U \longrightarrow (f(p), g(p)) \in \mathcal{S} \times \mathcal{S}$$

is a continuous mapping from U into the subset $\mathcal{S} \circ \mathcal{S} \subset \mathcal{S} \times \mathcal{S}$; the composition of $f \times g$ with the natural mapping $\mathcal{S} \circ \mathcal{S} \longrightarrow \mathcal{S}$, namely the mapping

$$f - g: p \in U \longrightarrow f(p) - g(p) \in \mathcal{S} ,$$

is therefore also a section. That is to say, the set $\Gamma(U, \mathcal{S})$ is a group, under the pointwise addition of sections. The <u>zero section</u>, the map $p \in U \longrightarrow 0_p \in \mathcal{S}_p$ where 0_p is the zero element of the group $\mathcal{S}_p$, is clearly a section. In general, it is rather difficult to determine whether there are any non-trivial sections, that is, any sections other than the zero section.

(b) In a sense, the sections of a sheaf determine the sheaf completely. This observation can be made more precise in the following manner.

<u>Definition</u>. A <u>presheaf</u> (of abelian groups) over a topological space M consists of:

(i) a basis $\{U_\alpha\}$ for the open sets of the topology of M ;

(ii) a separate abelian group $\mathcal{S}_\alpha$ associated to each open set U_α of the basis;

(iii) a homomorphism $\rho_{\alpha\beta}: \mathcal{S}_\beta \longrightarrow \mathcal{S}_\alpha$ associated to each inclusion relation $U_\alpha \subset U_\beta$, such that $\rho_{\alpha\beta}\rho_{\beta\gamma} = \rho_{\alpha\gamma}$ whenever $U_\alpha \subset U_\beta \subset U_\gamma$.

To each sheaf $\mathcal{S}$ over M and basis $\{U_\alpha\}$ for the open sets of the topology for M there is a naturally associated presheaf, which will be called the presheaf of sections of the sheaf $\mathcal{S}$; this is the presheaf which assigns to the set U_α the abelian group $\mathcal{S}_\alpha = \Gamma(U_\alpha, \mathcal{S})$, and assigns to the inclusion $U_\alpha \subset U_\beta$ the restriction mapping $\rho_{\alpha\beta}: \Gamma(U_\beta, \mathcal{S}) \longrightarrow \Gamma(U_\alpha, \mathcal{S})$ of sections over U_β to the subset U_α. Conversely, to any presheaf $\{U_\alpha, \mathcal{S}_\alpha, \rho_{\alpha\beta}\}$ over M there is an associated sheaf, which is constructed as follows. For each point $p \in M$ consider the collection

$$\mathcal{U}(p) = \{U_\alpha | p \in U_\alpha\} ;$$

this set is partially ordered under inclusion. Form the disjoint union $\mathcal{S}_p^* = \cup_{U_\alpha \in \mathcal{U}(p)} \mathcal{S}_\alpha$; and for any two elements $f_\alpha \in \mathcal{S}_\alpha$, $f_\beta \in \mathcal{S}_\beta$, write $f_\alpha \sim f_\beta$ if there exists a set $U_\gamma \in \mathcal{U}(p)$ such that $U_\gamma \subset U_\alpha \cap U_\beta$ and $\rho_{\gamma\alpha} f_\alpha = \rho_{\gamma\beta} f_\beta$. It is a straightforward exercise to verify that this is an equivalence relation. The set of equivalence classes in $\mathcal{S}_p^*$ will be denoted by $\mathcal{S}_p$. For any set U_α there is a natural mapping $\rho_{p\alpha}: \mathcal{S}_\alpha \longrightarrow \mathcal{S}_p$, which assigns to an element $f_\alpha \in \mathcal{S}_\alpha$ its equivalence class in $\mathcal{S}_p^*$. Again it is a simple matter to verify that these mappings $\rho_{p\alpha}$ induce on the set $\mathcal{S}_p$ the structure of an abelian group, in such a manner that the mappings $\rho_{p\alpha}$ are group homomorphisms. (The group $\mathcal{S}_p$ constructed from the family $\{\mathcal{S}_\alpha\}$ as above is called the direct limit group, $\mathcal{S}_p = \text{dir. lim.}_{U_\alpha \in \mathcal{U}(p)} \mathcal{S}_\alpha$; for a more general discussion of this concept, see S. Eilenberg and N. E. Steenrod, Foundations of Algebraic Topology, Chapter VIII, (Princeton University Press, 1952)). The space of the sheaf is defined to be the set

$$\mathcal{S} = \bigcup_{p \in M} \mathcal{S}_p ,$$

with the projection mapping $\pi: \mathcal{S} \longrightarrow M$ given by $\pi(\mathcal{S}_p) = p$. As a basis for the open sets to define the topology of $\mathcal{S}$ take sets of the form

$$[f_\alpha] = \bigcup_{p \in U_\alpha} \rho_{p\alpha}(f_\alpha) \subset \mathcal{S}$$

for the various elements $f_\alpha \in \mathcal{S}_\alpha$. (To see that these sets do form the basis for a topology, it is necessary to show that for any point $s \in [f_\alpha] \cap [f_\beta] \subset \mathcal{S}$ there is an element $f_\gamma \in \mathcal{S}_\gamma$ so that $s \in [f_\gamma] \subset [f_\alpha] \cap [f_\beta]$. Now if $s \in [f_\alpha] \cap [f_\beta]$, and if $p = \pi(s)$, then $p \in U_\alpha \cap U_\beta$ and $\rho_{p\alpha} f_\alpha = \rho_{p\beta} f_\beta$; by definition of the mapping $\rho_{p\alpha}$, there must be a set U_γ such that $p \in U_\gamma \subset U_\alpha \cap U_\beta$ and $\rho_{\gamma\alpha} f_\alpha = \rho_{\gamma\beta} f_\beta$. Therefore $s \in [f_\gamma] \subset [f_\alpha] \cap [f_\beta]$, as desired.) With this topology, it is clear that the mapping $\pi: \mathcal{S} \longrightarrow M$ is a local homeomorphism. Finally, to show that $\mathcal{S}$ is a sheaf, it merely remains to verify that the group operations are continuous. Select any point $(s_1, s_2) \in \mathcal{S} \circ \mathcal{S}$ and any open neighborhood $[f_\alpha]$ about $s_1 - s_2$; and let $p = \pi(s_1) = \pi(s_2)$. Further, select elements $f_{1\beta_1} \in \mathcal{S}_{\beta_1}$ and $f_{2\beta_2} \in \mathcal{S}_{\beta_2}$ such that $\rho_{p\beta_1}(f_{1\beta_1}) = s_1$ and $\rho_{p\beta_2}(f_{2\beta_2}) = s_2$. Then $\rho_{p\alpha}(f_\alpha) = \rho_{p\beta_1}(f_{1\beta_1}) - \rho_{p\beta_2}(f_{2\beta_2})$; so by the definition of the mapping $\rho_{p\alpha}$, there must be a set U_γ such that $\rho_{\gamma\alpha}(f_\alpha) = \rho_{\gamma\beta_1}(f_{1\beta_1}) - \rho_{\gamma\beta_2}(f_{2\beta_2})$. Now under the mapping $\mathcal{S} \circ \mathcal{S} \longrightarrow \mathcal{S}$ it follows that $([\rho_{\gamma\beta_1}(f_{1\beta_1})] \times [\rho_{\gamma\beta_2}(f_{2\beta_2})]) \cap \mathcal{S} \circ \mathcal{S}$ is an open neighborhood of (s_1, s_2) which maps into $[f_\alpha]$, proving the continuity.

Starting with a sheaf $\mathcal{S}$, form the presheaf of sections of $\mathcal{S}$ for some basis $\{U_\alpha\}$ for the topology of M. It is clear from the preceding construction that the associated sheaf of the presheaf of sections of $\mathcal{S}$ is canonically isomorphic to $\mathcal{S}$ itself. In this order, the two constructions introduced above are thus inverse to one another. It is not true, however, that these constructions are inverse to one another in the other order; that is, the presheaf of sections of the associated sheaf to a given presheaf is not always isomorphic to the given presheaf. For example, a presheaf in which $\mathcal{S}_\alpha \cong \mathbb{Z}$ for all α, and $\rho_{\alpha\beta}$ is the zero homomorphism, has the zero sheaf as its associated sheaf; and the presheaf of sections of the zero sheaf associates the zero group to each U_α. Clearly the problem is to characterize those presheaves which arise as the presheaf of sections of some sheaf.

<u>Definition.</u> A presheaf $\{U_\alpha, \mathcal{S}_\alpha, \rho_{\alpha\beta}\}$ over a topological space M is called a <u>complete presheaf</u> if, whenever $U_o = \bigcup_\beta U_\beta$ for a subcollection U_o, $\{U_\beta\}$ of the basis $\{U_\alpha\}$, the following two conditions are fulfilled:

(i) if $f_o, g_o \in \mathcal{S}_o$ are such that $\rho_{\beta o} f_o = \rho_{\beta o} g_o$ for all U_β, then $f_o = g_o$;

(ii) if $f_\beta \in \mathcal{S}_\beta$ are elements such that $\rho_{\gamma\beta_1} f_{\beta_1} = \rho_{\gamma\beta_2} f_{\beta_2}$ whenever $U_\gamma \subset U_{\beta_1} \cap U_{\beta_2}$ for any elements U_γ of the basis, there is an element $f_o \in \mathcal{S}_o$ such that $f_\beta = \rho_{\beta o} f_o$ for all U_β.

<u>Lemma 3.</u> A presheaf $\{U_\alpha, \mathcal{S}_\alpha, \rho_{\alpha\beta}\}$ over a topological space M is the presheaf of sections of some sheaf over M if and only if it is complete.

Proof. It is obvious that the presheaf of sections of any sheaf is complete; the converse assertion is the one of interest. Let $\mathcal{S}$ be the associated sheaf to the given presheaf. There is a canonical homomorphism $\rho: \mathcal{S}_\alpha \longrightarrow \Gamma(U_\alpha, \mathcal{S})$ defined by $\rho(f_\alpha) = [f_\alpha] = \bigcup_{p \in U_\alpha} \rho_{p\alpha}(f_\alpha)$. It suffices to prove that ρ is an isomorphism, for each set U_α. Suppose firstly that $\rho(f_\alpha) = 0$, that is, that $\rho_{p\alpha}(f_\alpha) = 0$ for all $p \in U_\alpha$. Then for each point $p \in U_\alpha$ there must exist a set U_β such that $p \in U_\beta \subset U_\alpha$ and $\rho_{\beta\alpha}(f_\alpha) = 0$. Since these sets $\{U_\beta\}$ cover U_α, it follows from property (i) of a complete presheaf that $f_\alpha = 0$. Next, consider any section $f \in \Gamma(U_\alpha, \mathcal{S})$. For each point $p \in U_\alpha$ there must exist a set U_β with $p \in U_\beta \subset U_\alpha$, and an element $f_\beta \in \mathcal{S}_\beta$ such that $\rho_{p\beta}(f_\beta) = f(p)$. The sections f and $[f_\beta]$ coincide at p, hence in a full open neighborhood of p; so by restricting U_β further if necessary, $\rho_{q\beta}(f_\beta) = f(q)$ for all points $q \in U_\beta$. The ~~sets~~ $\{U_\beta\}$ cover U_α, and obviously satisfy condition (ii) of the definition of a complete presheaf; therefore there is some element $f_\alpha \in \mathcal{S}_\alpha$ such that $\rho_{\beta\alpha}(f_\alpha) = f_\beta$, hence such that $f = [f_\alpha]$. This suffices to complete the proof.

(c) Let M be a Riemann surface, and $\{U_\alpha\}$ be any basis for the open sets in the topology of M. To each set U_α associate the ring $\mathcal{O}_{U_\alpha}$ of functions holomorphic in U_α; and to each inclusion relation $U_\alpha \subset U_\beta$ associate the natural restriction mapping $\rho_{\alpha\beta}: \mathcal{O}_{U_\beta} \longrightarrow \mathcal{O}_{U_\alpha}$. Clearly $\{U_\alpha, \mathcal{O}_{U_\alpha}, \rho_{\alpha\beta}\}$ is a presheaf over M; the associated sheaf is called the sheaf of germs of holomorphic functions on M, and will be denoted by $\mathcal{O}$. (For the purposes at hand, it is the additive structure of the rings $\mathcal{O}_{U_\alpha}$ which will be considered, to obtain a sheaf of abelian groups. Actually, of course, $\mathcal{O}$ is a sheaf of rings over the space M, using the obvious modifications of the preceding definitions. The ring structure plays a very important role in the case of holomorphic functions of several complex variables; for more in this direction, see R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Chapter IV, (Prentice-Hall, 1965).)

To interpret the stalk $\mathcal{O}_p$ of the sheaf $\mathcal{O}$ at a point $p \in M$, select a coordinate neighborhood U containing p and a coordinate mapping $z: U \longrightarrow V \subset \mathbb{C}$ such that $z(p) = 0$. The construction of the stalk $\mathcal{O}_p$ being local, it is sufficient to consider the entire construction within the set $V \subset \mathbb{C}$. To each open neighborhood of 0 in V consider the ring of functions holomorphic in that neighborhood. Two such functions are equivalent if they agree in some smaller open neighborhood of 0; and the equivalence classes, called the <u>germs of holomorphic functions</u> at 0, form the stalk $\mathcal{O}_p$. To each function holomorphic in an open neighborhood of 0 associate its power series expansion at the point 0; equivalent functions clearly determine the same power series, and every convergent power series arises from some unique germ. Therefore $\mathcal{O}_p$ is isomorphic to the ring $\mathbb{C}\{z\}$ of convergent, complex power series in the variable z.

Note that the presheaf $\{U_\alpha, \mathcal{O}_{U_\alpha}, \rho_{\alpha\beta}\}$ is obviously complete; hence with the natural isomorphism it is possible to identify $\Gamma(U_\alpha, \mathcal{O}) = \mathcal{O}_{U_\alpha}$. That is, the sections of the sheaf $\mathcal{O}$ over any open set $U \subset M$ are identified with the ring $\mathcal{O}_U$ of functions holomorphic in U. A similar construction, beginning with the multiplicative groups $\mathcal{O}_U^*$ of nowhere-vanishing holomorphic functions on U leads to the sheaf $\mathcal{O}^*$ of germs of nowhere-vanishing holomorphic functions; again, in the same manner, it is possible to introduce the sheaf $\mathcal{M}$ of germs of meromorphic functions on M, (a sheaf of fields, actually), or the multiplicative sheaf $\mathcal{M}^*$ of germs of meromorphic functions not identically zero on M.

Considering merely the differentiable structure of M, and the rings $\mathcal{C}_U^\infty$ or $\mathcal{C}_U$ of infinitely differentiable or continuous functions on open sets $U \subset M$, leads in a similar manner to the sheaf $\mathcal{C}^\infty$ of germs of C^∞ functions on M, or the sheaf $\mathcal{C}$ of germs of continuous functions on M. Note that there are the natural inclusion relations

$$\mathcal{O} \subset \mathcal{C}^\infty \subset \mathcal{C} .$$

(d) Various relations between sheaves over a fixed topological space M are of importance in the applications. First, for a sheaf $\mathcal{S}$ over M, let $E \subset M$ be an arbitrary subset of M. The <u>restriction</u> of the sheaf $\mathcal{S}$ to E is the subset $\pi^{-1}(E) \subset \mathcal{S}$, where $\pi\colon \mathcal{S} \to M$ is the projection mapping; the restriction, which will be denoted by $\mathcal{S}|E$, is clearly a sheaf over the set E. In particular, for a point $p \in M$, $\mathcal{S}|p = \mathcal{S}_p$ is just the stalk of $\mathcal{S}$ over p. (For example, if M is a Riemann surface, then an open subset $U \subset M$ is also a Riemann surface, and its sheaf of germs of holomorphic functions is just $\mathcal{O}|U$, the restriction of the sheaf $\mathcal{O}$ over M to the subset U. If $E \subset M$ is not an open subset, the sheaf $\mathcal{O}|E$ cannot generally be interpreted as a subsheaf of the sheaf of germs of continuous functions on the space E itself; the case in which E is a single point of M illustrates why.)

Again let $\mathcal{S}$ be a sheaf of abelian groups over M, and let $\mathcal{R} \subset \mathcal{S}$ be a subset of $\mathcal{S}$. Then $\mathcal{R}$ is called a subsheaf of $\mathcal{S}$ if:

(i) $\mathcal{R}$ is an open subset of $\mathcal{S}$; and

(ii) for each point $p \in M$, $\mathcal{R}_p = \mathcal{R} \cap \mathcal{S}_p$ is a subgroup of $\mathcal{S}_p$.

Clearly $\mathcal{R}$ is itself a sheaf of abelian groups over M, its projection mapping being the restriction to $\mathcal{R}$ of the projection mapping π of $\mathcal{S}$. The quotient sheaf $\mathcal{J} = \mathcal{S}/\mathcal{R}$ is then defined as follows. For each point $p \in M$ let $\mathcal{J}_p = \mathcal{S}_p/\mathcal{R}_p$ be the natural quotient group; and let $\mathcal{J} = \bigcup_{p \in M} \mathcal{J}_p$, with the projection mapping $\pi: \mathcal{J} \longrightarrow M$ given by $\pi(\mathcal{J}_p) = p$. The natural mapping $\varphi: \mathcal{S} \longrightarrow \mathcal{J}$ which associates to any element of $\mathcal{S}_p$ its coset in $\mathcal{J}_p = \mathcal{S}_p/\mathcal{R}_p$ commutes with the projections π in $\mathcal{S}$ and in $\mathcal{J}$. Introduce on $\mathcal{J}$ the natural quotient topology, defining a set $U \subset \mathcal{J}$ to be open if and only if $\varphi^{-1}(U)$ is open in $\mathcal{S}$. It is an easy matter, which will be left to the reader, to verify that $\mathcal{J}$ is then a sheaf.

For example let M be a Riemann surface, and $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M, as usual. Let $P = \{p_1,\dots,p_n\}$ be a finite number of distinct points on M, and for each open subset $U \subset M$ consider the set

$$\mathcal{R}_U = \{f \in \mathcal{O}_U | f(p_i) = 0 \text{ whenever } p_i \in U,\ i = 1,\dots,n\}$$

Note that each $\mathcal{R}_U$ is a subgroup of $\mathcal{O}_U$; and that the groups $\{\mathcal{R}_U\}$ for all the open subsets of M, with the natural restriction mappings, form a presheaf over M. The associated sheaf is then a subsheaf

$\mathcal{R} \subset \mathcal{O}$, and this leads to a quotient sheaf $\mathcal{J} = \mathcal{O}/\mathcal{R}$. To describe this quotient sheaf, note that for a point $p \notin P$ necessarily $\mathcal{R}_p = \mathcal{O}_p$, hence $\mathcal{J}_p = 0$. However for a point $p_1 \in P$, $\mathcal{R}_{p_1} \subset \mathcal{O}_{p_1}$ is the subgroup consisting of those germs of holomorphic functions which vanish at p_1 , or equivalently, the subgroup of those convergent power series with zero constant term; thus $\mathcal{J}_{p_1} = \mathcal{O}_{p_1}/\mathcal{R}_{p_1} \cong \mathbb{C}$, the latter isomorphism being that which associates to any power series in $\mathcal{O}_{p_1}$ its constant term. That is to say, the sheaf $\mathcal{J}$ will have a trivial stalk (consisting of the zero group alone) at all points $p \notin P$, and it will have stalk $\mathcal{J}_p \cong \mathbb{C}$ at the points $p \in P$. A sheaf of this sort is sometimes called a skyscraper sheaf.

Now suppose that $\mathcal{S}$ and $\mathcal{J}$ are two sheaves of abelian groups over M , with projection mappings $\sigma: \mathcal{S} \longrightarrow M$ and $\tau: \mathcal{J} \longrightarrow M$. A mapping $\varphi: \mathcal{S} \longrightarrow \mathcal{J}$ is called a <u>sheaf mapping</u> if: (i) φ is continuous; and (ii) $\tau \circ \varphi = \sigma$. The second condition implies that, for any point $p \in M$, $\varphi(\mathcal{S}_p) \subset \mathcal{J}_p$; so a sheaf mapping preserves stalks. Further, for any $f \in \Gamma(U, \mathcal{S})$ for an open subset $U \subset M$, $\varphi \circ f$ will be a continuous mapping from U into $\mathcal{J}$ such that $\tau \circ (\varphi \circ f) = \sigma \circ f =$ the identity; that is, $\varphi \circ f \in \Gamma(U, \mathcal{J})$. Therefore the sheaf mapping φ yields an <u>induced mapping</u> $\varphi^*: \Gamma(U, \mathcal{S}) \longrightarrow \Gamma(U, \mathcal{J})$. In particular, since $\{f(U)\}$, for all open subsets $U \subset M$ and all sections $f \in \Gamma(U, \mathcal{S})$, is a basis for the topology of $\mathcal{S}$, the sheaf mapping is open as well as continuous; and since the mappings σ and τ are local homeomorphisms, so is φ. That is to say, any sheaf mapping is necessarily a local homeomorphism between the spaces $\mathcal{S}$ and $\mathcal{J}$. The sheaf mapping φ is called a sheaf homomorphism if it is a homomorphism on each stalk; the induced mapping φ^* is then a homomorphism between the groups of sections, called the induced homomorphism. A sheaf isomorphism is a sheaf homomorphism with an inverse which is also a sheaf homomorphism; the notation $\mathcal{S} \cong \mathcal{J}$ will be used to indicate that the sheaves $\mathcal{S}$ and $\mathcal{J}$ over M are isomorphic.

For example, considering again the sheaf $\mathcal{O}$ over a Riemann surface M, to each germ $f_p \in \mathcal{O}_p$ associate the germ $e(f_p) = \exp.(2\pi i\, f_p) \in \mathcal{O}_p^*$. This determines a sheaf homomorphism $e: \mathcal{O} \longrightarrow \mathcal{O}^*$. Similarly of course, considering merely the sheaves of germs of continuous rather than holomorphic functions, there is a sheaf homomorphism $e: \mathcal{C} \longrightarrow \mathcal{C}^*$.

For any sheaf homomorphism $\varphi: \mathcal{S} \longrightarrow \mathcal{J}$ over a space M, the kernel of φ is the subset of $\mathcal{S}$ consisting of those points which map into the zero element of any group $\mathcal{J}_p$; that is, the kernel is the subset $\varphi^{-1}(0) \subset \mathcal{S}$ where $0 \in \Gamma(M, \mathcal{J})$ is the zero section of $\mathcal{J}$. Since the zero section is an open subset of $\mathcal{J}$, the kernel is clearly a subsheaf of $\mathcal{S}$. The image of φ is a subsheaf of $\mathcal{J}$ as well; and it follows readily that $\text{image}(\varphi) \cong \mathcal{S}/\text{kernel}(\varphi)$, where $\cong$ denotes isomorphism of sheaves. Given sheaf homomorphisms $\varphi: \mathcal{R} \longrightarrow \mathcal{S}$ and $\psi: \mathcal{S} \longrightarrow \mathcal{J}$, the diagram

$$\mathcal{R} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\psi} \mathcal{J}$$

will be called an exact sequence of sheaves if the image of φ is precisely the kernel of ψ. Similarly, a longer string of sheaves and sheaf homomorphisms will be called an exact sequence if for any two consecutive homomorphisms, the image of the one is precisely the kernel of the other. In particular, if 0 denotes the trivial sheaf with stalk the zero group at each point of M , a sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\psi} \mathcal{J} \longrightarrow 0$$

is exact if and only if φ is an injection, (an isomorphism from $\mathcal{R}$ to a subsheaf of $\mathcal{S}$), ψ is a projection, (a homomorphism with image all of $\mathcal{J}$), and the kernel of ψ precisely the image of φ ; hence, $\mathcal{J} \cong \mathcal{S}/\varphi(\mathcal{R})$. Conversely, if $\mathcal{R}$ is a subsheaf of $\mathcal{S}$ the inclusion mapping $i: \mathcal{R} \longrightarrow \mathcal{S}$ is a sheaf homomorphism; and the natural mapping $\varphi: \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{R}$ is a sheaf homomorphism, such that

$$0 \longrightarrow \mathcal{R} \xrightarrow{i} \mathcal{S} \xrightarrow{\varphi} \mathcal{S}/\mathcal{R} \longrightarrow 0$$

is an exact sequence of sheaves.

For example, on a Riemann surface M the subset $\mathbb{Z} \subset \mathcal{O}$ of germs of holomorphic functions which take only integer values is a subsheaf of $\mathcal{O}$ isomorphic to the constant sheaf; and this is precisely the kernel of the sheaf homomorphism $e: \mathcal{O} \longrightarrow \mathcal{O}^*$ introduced above. This homomorphism e is a projection, since any germ $f_p \in \mathcal{O}_p^*$ has a holomorphic logarithm near p . Therefore there arises the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0 .$$

Similarly, considering the sheaves of germs of continuous functions, there is the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C} \xrightarrow{e} \mathcal{C}^* \longrightarrow 0 .$$

§3. Cohomology

(a) Let M be a topological space, and $\mathcal{U} = \{U_\alpha\}$ be an open covering of M. To this covering of M there is associated a simplicial complex $N(\mathcal{U})$, called the nerve of the covering $\mathcal{U}$, and defined as follows. The vertices of $N(\mathcal{U})$ are the sets U_α of the covering. Vertices $U_0, \ldots, U_q$ span a q-simplex $\sigma = (U_0, \ldots, U_q)$ if and only if $U_0 \cap \ldots \cap U_q \neq \emptyset$; the set $U_0 \cap \ldots \cap U_q = |\sigma|$ is called the support of the simplex σ. Let $\mathcal{S}$ be a sheaf of abelian groups over M. A q-cochain of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$ is a function f which associates to every q-simplex $\sigma \in N(\mathcal{U})$ a section $f(\sigma) \in \Gamma(|\sigma|, \mathcal{S})$; the set of all such q-cochains will be denoted by $C^q(\mathcal{U}, \mathcal{S})$. Whenever $f, g \in C^q(\mathcal{U}, \mathcal{S})$, their sum $f + g \in C^q(\mathcal{U}, \mathcal{S})$, where $(f+g)(\sigma) = f(\sigma) + g(\sigma)$; so $C^q(\mathcal{U}, \mathcal{S})$ is an abelian group. There is also an operator $\delta\colon C^q(\mathcal{U}, \mathcal{S}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{S})$ called the coboundary operator, defined as follows. If $f \in C^q(\mathcal{U}, \mathcal{S})$ and $\sigma = (U_0, \ldots, U_{q+1}) \in N(\mathcal{U})$, then

$$(1) \qquad (\delta f)(U_0, \ldots, U_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|} f(U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{q+1}),$$

where $\rho_{|\sigma|}$ denotes the restriction of the section $f(U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{q+1}) \in \Gamma(U_0 \cap \ldots \cap U_{i-1} \cap U_{i+1} \cap \ldots \cap U_{q+1})$ to $|\sigma| = U_0 \cap \ldots \cap U_{q+1}$. It is clear that δ is a group homomorphism; and it is a straightforward calculation to show that $\delta\delta = 0$. The subset $Z^q(\mathcal{U}, \mathcal{S}) = \{f \in C^q(\mathcal{U}, \mathcal{S}) \mid \delta f = 0\}$ is a subgroup of $C^q(\mathcal{U}, \mathcal{S})$ called the group of q-cocycles; the image $\delta C^{q-1}(\mathcal{U}, \mathcal{S}) \subset C^q(\mathcal{U}, \mathcal{S})$ is called the group of q-coboundaries, and is a subgroup of the group of cocycles since $\delta\delta = 0$. The quotient group

$$H^q(\mathcal{U}, \mathcal{S}) = \begin{cases} Z^q(\mathcal{U}, \mathcal{S})/\delta C^{q-1}(\mathcal{U}, \mathcal{S}) & \text{for } q > 0 \\ Z^0(\mathcal{U}, \mathcal{S}) & \text{for } q = 0 \end{cases}$$

is called the <u>q-th cohomology group of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$.</u>

<u>Lemma 4.</u> $H^0(\mathcal{U}, \mathcal{S}) \cong \Gamma(M, \mathcal{S})$.

Proof. By the above definition, $H^0(\mathcal{U}, \mathcal{S}) = Z^0(\mathcal{U}, \mathcal{S})$. A zero-cochain $f \in C^0(\mathcal{U}, \mathcal{S})$ is a function which assigns to each set $U \in \mathcal{U}$ a section $f(U) \in \Gamma(U, \mathcal{S})$; and its coboundary $\delta f \in C^1(\mathcal{U}, \mathcal{S})$ is a function which assigns to each pair of intersecting sets $U_0, U_1 \in \mathcal{U}$ a section $(\delta f)(U_0, U_1) = \rho_{U_0 \cap U_1} f(U_1) - \rho_{U_0 \cap U_1} f(U_1) \in \Gamma(U_0 \cap U_1, \mathcal{S})$. If $\delta f = 0$, the sections $f(U_\alpha)$ agree in each non-empty intersection $U_0 \cap U_1$, hence altogether determine a section of $\mathcal{S}$ defined over the entire space M; and conversely, the zero-cochain defined by restricting a global section of $\mathcal{S}$ over M to the various subsets U_α is a zero-cocycle. This suffices to prove the desired result.

(b) In order to have a cohomology theory associated intrinsically to the space M, it is necessary to consider various possible coverings of M. A covering $\mathcal{V} = \{V_a\}$ is called a <u>refinement</u> of the covering $\mathcal{U} = \{U_\alpha\}$ if there is a mapping $\mu: \mathcal{V} \longrightarrow \mathcal{U}$ such that $V_a \subset \mu V_a$ for each $V_a \in \mathcal{V}$; the mapping μ is called a <u>refining mapping.</u> The covering $\mathcal{V}$ may of course be a refinement of $\mathcal{U}$ by various different refining mappings. Notice that the refining mapping induces a mapping

$$\mu: C^q(\mathcal{U}, \mathcal{S}) \longrightarrow C^q(\mathcal{V}, \mathcal{S}),$$

as follows. If $f \in C^q(\mathcal{U}, \mathcal{S})$ and $\sigma = (V_0, \ldots, V_q) \in N(\mathcal{V})$, then $(\mu f)(V_0, \ldots, V_q) = \rho_{|\sigma|} f(\mu V_0, \ldots, \mu V_q)$; since $\mu V_0 \cap \ldots \cap \mu V_q \supset V_0 \cap \ldots \cap V_q \neq \emptyset$, then $(\mu V_0, \ldots, \mu V_q)$ is a q-simplex of $N(\mathcal{U})$, and the mapping is thus well-defined. It is clear that μ is a group homomorphism, and that $\mu\delta = \delta\mu$; therefore μ determines a homomorphism

$$\mu^*: H^q(\mathcal{U}, \mathcal{S}) \longrightarrow H^q(\mathcal{V}, \mathcal{S}).$$

Lemma 5. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, and if $\mu: \mathcal{V} \longrightarrow \mathcal{U}$ and $\nu: \mathcal{V} \longrightarrow \mathcal{U}$ are two refining mappings, then $\mu^* = \nu^*$.

Proof. When $q = 0$, the mappings $\mu^*: H^0(\mathcal{U}, \mathcal{S}) \longrightarrow H^0(\mathcal{V}, \mathcal{S})$ and $\nu^*: H^0(\mathcal{U}, \mathcal{S}) \longrightarrow H^0(\mathcal{V}, \mathcal{S})$ are both the identity mapping, in view of Lemma 4; so it is only necessary to consider the case $q > 0$. In this case, construct an associated $\theta: C^q(\mathcal{U}, \mathcal{S}) \longrightarrow C^{q-1}(\mathcal{V}, \mathcal{S})$ as follows. If $f \in C^q(\mathcal{U}, \mathcal{S})$ and $\sigma = (V_0, \ldots, V_{q-1}) \in N(\mathcal{V})$, define $(\theta f)(V_0, \ldots, V_{q-1}) = \sum_{j=0}^{q-1} (-1)^j \rho_{|\sigma|} f(\mu V_0, \ldots, \mu V_j, \nu V_j, \ldots, \nu V_{q-1})$. Now whenever $\sigma = (V_0, \ldots, V_q) \in N(\mathcal{V})$, observe that

$$\begin{aligned}
&(\delta\theta f)(V_0, \ldots, V_q) \\
&= \sum_{j=0}^{q} (-1)^{j+1} \rho \Big\{ \sum_{i=0}^{j-1} (-1)^i \rho f(\mu V_0, \ldots, \mu V_{i-1}, \mu V_{i+1}, \ldots, \mu V_j, \nu V_j, \ldots, \nu V_q) \\
&\qquad + \sum_{i=j+1}^{q} (-1)^{i+1} \rho f(\mu V_0, \ldots, \mu V_j, \nu V_j, \ldots, \nu V_{i-1}, \nu V_{i+1}, \ldots, \nu V_q) \Big\} \\
&= \sum_{j=0}^{q} (-1)^{j+1} \rho\delta f(\mu V_0, \ldots, \mu V_j, \nu V_j, \ldots, \nu V_q) + \nu f(\sigma) - \mu f(\sigma).
\end{aligned}$$

Therefore, if $f \in Z^q(\mathfrak{U}, \mathcal{S})$ so that $\delta f = 0$, it follows that $\nu f - \mu f = \delta\theta f$, that is, that νf and μf determine the same cohomology class. Therefore $\mu^* = \nu^*$, as desired.

Now for any two coverings $\mathfrak{U}$, $\mathfrak{V}$ of M, write $\mathfrak{V} < \mathfrak{U}$ if $\mathfrak{V}$ is a refinement of $\mathfrak{U}$; the set of all coverings is partially ordered under this relation; and by Lemma 5 there is a unique homomorphism $H^q(\mathfrak{U}, \mathcal{S}) \longrightarrow H^q(\mathfrak{V}, \mathcal{S})$ whenever $\mathfrak{V} < \mathfrak{U}$. It is clear that these homomorphisms are transitive; hence it is possible to introduce the direct limit group

$$H^q(M, \mathcal{S}) = \text{dir.lim.}_{\mathfrak{U}}\ H^q(\mathfrak{U}, \mathcal{S}),$$

which will be called the <u>q-th cohomology group of the space M with coefficients in the sheaf $\mathcal{S}$.</u> (Recall that to define the direct limit, introduce the disjoint union $\cup_{\mathfrak{U}} H^q(\mathfrak{U}, \mathcal{S})$; and for cohomology classes $f \in H^q(\mathfrak{U}, \mathcal{S})$, $g \in H^q(\mathfrak{V}, \mathcal{S})$, write $f \sim g$ if there is a refinement $\mathfrak{W} < \mathfrak{U}$ and $\mathfrak{W} < \mathfrak{V}$, such that f and g have the same image under the natural homomorphisms $H^q(\mathfrak{U}, \mathcal{S}) \longrightarrow H^q(\mathfrak{W}, \mathcal{S})$ and $H^q(\mathfrak{V}, \mathcal{S}) \longrightarrow H^q(\mathfrak{W}, \mathcal{S})$. This relation is an equivalence relation, and the set of equivalence classes in the direct limit group $H^q(M, \mathcal{S})$.) For each covering $\mathfrak{U}$, there is the natural homomorphism $H^q(\mathfrak{U}, \mathcal{S}) \longrightarrow H^q(M, \mathcal{S})$. It follows immediately from Lemma 4 that

$$H^0(M, \mathcal{S}) \cong \Gamma(M, \mathcal{S}). \tag{2}$$

It should be noted that for a constant sheaf such as $\mathbf{Z}$, the cohomology introduced above coincides with the ordinary Čech cohomology with coefficients in the group $\mathbf{Z}$; (see S. Eilenberg and N. E. Steenrod, Foundations of Algebraic Topology, Chapter IX, (Princeton University Press, 1952).)

(c) Consider an exact sequence of sheaves of the following form, over the space M :

$$0 \longrightarrow \mathcal{R} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\psi} \mathcal{J} \longrightarrow 0 .$$

For any open subset $U \subset M$, the sheaf homomorphisms φ, ψ induce homomorphisms φ^*, ψ^* between the corresponding groups of sections, and there results an exact sequence of groups and homomorphisms of the form

$$0 \longrightarrow \Gamma(U, \mathcal{R}) \xrightarrow{\varphi^*} \Gamma(U, \mathcal{S}) \xrightarrow{\psi^*} \Gamma(U, \mathcal{J}) .$$

Exactness this far is obvious, since φ can be considered as an injection of $\mathcal{R}$ as a subsheaf of $\mathcal{S}$, and ψ as the passage to the quotient sheaf; but in general the mapping ψ^* will not have all of $\Gamma(U, \mathcal{J})$ as its image. (For instance, let M be the annulus $1 < |z| < 2$ in $\mathbf{C}$, and consider over M the exact sequence $0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0$ introduced in §2(d); the function $z \in \Gamma(M, \mathcal{O}^*)$ cannot be written in the form $z = \exp. 2\pi i\, f(z)$ for any $f(z) \in \Gamma(M, \mathcal{O})$, since necessarily $f(z) = \frac{1}{2\pi i} \log z$ and no branch of $\log z$ is a single-valued holomorphic function on M.) The cohomology theory considered above furnishes a convenient measure of the extent of the inexactness of the sequence of sections. However, it is necessary to have further restrictions on the underlying topological space M.

(Recall that an open covering $\mathcal{U} = \{U_\alpha\}$ of a topological space M is called locally finite if each point $p \in M$ has an open neighborhood V which meets at most finitely many of the sets U_α.

A Hausdorff space M is called paracompact if every open covering has a locally finite refinement. Any separable manifold is paracompact. In defining the cohomology groups of a paracompact space, it is sufficient to consider merely the locally finite coverings $\mathcal{U}$, rather than all open coverings, in the direct limit construction introduced above. For further discussion of the topological properties, see for instance J. L. Kelley, General Topology, (Van Nostrand, 1955).)

Theorem 1. If M is a paracompact Hausdorff space, and if

$$0 \longrightarrow \mathcal{R} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\psi} \mathcal{J} \longrightarrow 0$$

is an exact sequence of sheaves of abelian groups over M, then there is an exact sequence of cohomology groups of the form

$$0 \longrightarrow H^0(M, \mathcal{R}) \xrightarrow{\varphi^*} H^0(M, \mathcal{S}) \xrightarrow{\psi^*} H^0(M, \mathcal{J}) \xrightarrow{\delta^*} H^1(M, \mathcal{R}) \xrightarrow{\varphi^*}$$

$$H^1(M, \mathcal{S}) \xrightarrow{\psi^*} H^1(M, \mathcal{J}) \xrightarrow{\delta^*} H^2(M, \mathcal{R}) \longrightarrow \dots .$$

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open covering of the space M. For each simplex $\sigma \in N(\mathcal{U})$ there is an induced exact sequence $0 \longrightarrow \Gamma(|\sigma|, \mathcal{R}) \xrightarrow{\varphi} \Gamma(|\sigma|, \mathcal{S}) \xrightarrow{\psi} \Gamma(|\sigma|, \mathcal{J})$; and since the cochain groups are merely direct sums of the groups $\Gamma(|\sigma|, *)$, there follow the exact sequences of cochain groups $0 \longrightarrow C^q(\mathcal{U}, \mathcal{R}) \xrightarrow{\varphi} C^q(\mathcal{U}, \mathcal{S}) \xrightarrow{\psi} C^q(\mathcal{U}, \mathcal{J})$. Defining $\overline{C}^q(\mathcal{U}, \mathcal{J}) = \psi C^q(\mathcal{U}, \mathcal{S}) \subset C^q(\mathcal{U}, \mathcal{J})$, these sequences can be extended to full exact sequences of the form $0 \longrightarrow C^q(\mathcal{U}, \mathcal{R}) \xrightarrow{\varphi} C^q(\mathcal{U}, \mathcal{S}) \xrightarrow{\psi} \overline{C}^q(\mathcal{U}, \mathcal{J}) \longrightarrow 0$. The homomorphisms φ and ψ clearly commute with the coboundary mappings, in the sense that $\varphi\delta = \delta\varphi$ and $\psi\delta = \delta\psi$; there results an extensive commutative diagram of the following sort, in which all the rows are exact sequences of groups.

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{q-1}(\mathcal{U},\mathcal{R}) & \xrightarrow{\varphi} & C^{q-1}(\mathcal{U},\mathcal{S}) & \xrightarrow{\psi} & \overline{C}^{q-1}(\mathcal{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
0 & \longrightarrow & C^{q}(\mathcal{U},\mathcal{R}) & \xrightarrow{\varphi} & C^{q}(\mathcal{U},\mathcal{S}) & \xrightarrow{\psi} & \overline{C}^{q}(\mathcal{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
0 & \longrightarrow & C^{q+1}(\mathcal{U},\mathcal{R}) & \xrightarrow{\varphi} & C^{q+1}(\mathcal{U},\mathcal{S}) & \xrightarrow{\psi} & \overline{C}^{q+1}(\mathcal{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

Now it follows immediately from an examination of this diagram that for each index q there is an exact sequence of cohomology groups $H^q(\mathcal{U},\mathcal{R}) \xrightarrow{\varphi^*} H^q(\mathcal{U},\mathcal{S}) \xrightarrow{\psi^*} \overline{H}^q(\mathcal{U},\mathcal{T})$, where by definition $\overline{H}^q(\mathcal{U},\mathcal{T}) = \overline{Z}^q(\mathcal{U},\mathcal{T})/\delta\overline{C}^{q-1}(\mathcal{U},\mathcal{T})$ and $\overline{Z}^q(\mathcal{U},\mathcal{T}) = \{f \in \overline{C}^q(\mathcal{U},\mathcal{T}) \mid \delta f = 0\}$. Mappings $\delta^*: \overline{H}^q(\mathcal{U},\mathcal{T}) \longrightarrow H^{q+1}(\mathcal{U},\mathcal{R})$ are then constructed in this manner. If $f \in \overline{C}^q(\mathcal{U},\mathcal{T})$ is an element for which $\delta f = 0$, select an element $g \in C^q(\mathcal{U},\mathcal{S})$ such that $\psi g = f$; then since $\psi\delta g = \delta\psi g = \delta f = 0$, by exactness there must exist an element $h \in C^{q+1}(\mathcal{U},\mathcal{R})$ such that $\varphi h = \delta g$. Define $\delta^*[f] = [h]$, where $[f] \in \overline{H}^q(\mathcal{U},\mathcal{T})$ is the cohomology class of f and $[h] \in H^{q+1}(\mathcal{U},\mathcal{R})$ is the cohomology class of h.. Of course, it is necessary to observe that $\delta h = 0$, and that $[h]$ is independent of the choices made in this construction, namely, the choice of representative f in the cohomology class $[f]$

and the choice of the element g ; this is straightforward, and will be left to the reader. Finally, another simple diagram chase, which will also be left to the reader, shows that the resulting cohomology sequence $\cdots \longrightarrow H^q(\mathcal{U}, \mathcal{R}) \xrightarrow{\varphi^*} H^q(\mathcal{U}, \mathcal{S}) \xrightarrow{\psi^*} \bar{H}^q(\mathcal{U}, \mathcal{D}) \xrightarrow{\delta^*} H^{q+1}(\mathcal{U}, \mathcal{R}) \xrightarrow{\varphi^*} H^{q+1}(\mathcal{U}, \mathcal{S}) \longrightarrow \cdots$ is an exact sequence.

Next, consider a refinement $\mu\colon \mathcal{V} \longrightarrow \mathcal{U}$ of the covering $\mathcal{U}$. There is a similar exact cohomology sequence for the covering $\mathcal{V}$; and it is easy to see that the induced cohomology homomorphism μ^* commutes with the homomorphisms φ^*, ψ^*, δ^* of each cohomology sequence. (In particular, $\mu^*\bar{H}^q(\mathcal{U}, \mathcal{D}) \subset \bar{H}^q(\mathcal{V}, \mathcal{D})$.) Upon passing to the direct limit, there then follows an exact cohomology sequence for the space M,

$$\cdots \longrightarrow H^q(M, \mathcal{R}) \xrightarrow{\varphi^*} H^q(M, \mathcal{S}) \xrightarrow{\psi^*} \bar{H}^q(M, \mathcal{D}) \xrightarrow{\delta^*} H^{q+1}(M, \mathcal{R}) \longrightarrow \cdots$$

Up to this point, the regularity properties of the space M have not been required; but the cohomology groups $\bar{H}^q(M, \mathcal{D})$ remain to be investigated. It will next be demonstrated that, for a paracompact Hausdorff space M, $\bar{H}^q(M, \mathcal{D}) = H^q(M, \mathcal{D})$, which will suffice to complete the proof. It clearly suffices merely to show the following: given a cochain $f \in C^q(\mathcal{U}, \mathcal{D})$, there exist a refinement $\mu\colon \mathcal{V} \longrightarrow \mathcal{U}$ and a cochain $g \in C^q(\mathcal{V}, \mathcal{S})$, such that $\mu f = \psi g$. Since M is paracompact Hausdorff, and hence normal, there are open sets W_α such that $\bar{W}_\alpha \subset U_\alpha$ and the W_α cover M ; and the covering $\mathcal{U}$ can be assumed to be locally finite. For each point $p \in M$, select an open neighborhood V_p of p sufficiently small that:

(i) $V_p \subset W_\alpha$ for at least one set W_α ;

(ii) if $V_p \cap W_\alpha \neq \emptyset$ then $V_p \subset U_\alpha$; and

(iii) if $\sigma = (U_0,\dots,U_q) \in N(\mathcal{U})$ and $p \in |\sigma|$, (so necessarily $V_p \subset |\sigma|$) , then $\rho_{V_p} f(\sigma)$ is the image under ψ of a section of $\mathcal{S}$ over V_p .

For each set V_p select a set $\mu(V_p) = U_p \in \mathcal{U}$ such that $V_p \subset W_p \subset U_p$; this is always possible by (i), and then exhibits the set $\mathcal{V} = \{V_p\}$ as a refinement of $\mathcal{U}$. For any q-simplex $\sigma = (V_{p_0},\dots,V_{p_q}) \in N(\mathcal{V})$, note that $\emptyset \neq |\sigma| = V_{p_0} \cap \dots \cap V_{p_q} \subset W_{p_0} \cap \dots \cap W_{p_q}$; since $V_{p_0} \cap W_{p_i} \neq \emptyset$, it follows from (ii) that $V_{p_0} \subset U_{p_i}$ for each i , hence that $|\sigma| \subset V_{p_0} \subset U_{p_0} \cap \dots \cap U_{p_q} = |\mu\sigma|$. Therefore $\mu f(\sigma) = \rho_{|\sigma|} f(\mu V_{p_0},\dots,\mu V_{p_q}) = \rho_{|\sigma|}\rho_{V_{p_0}} f(U_{p_0},\dots,U_{p_q})$. However it follows from (iii) that the restriction to V_{p_0} of the section $f(\mu\sigma)$ already lies in the image of ψ , and this suffices to conclude the proof.

(d) Let $\mathcal{S}$ be a sheaf of abelian groups over the topological space M , and let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open covering of M . A <u>partition of unity</u> for the sheaf $\mathcal{S}$ subordinate to the covering $\mathcal{U}$ is a family of sheaf homomorphisms $\eta_\alpha : \mathcal{S} \longrightarrow \mathcal{S}$ such that:

(i) $\eta_\alpha(\mathcal{S}_p) = 0$ for all $p \in M - U_\alpha$;

(ii) $\Sigma_\alpha\, \eta_\alpha(s) = s$ for any $s \in \mathcal{S}$.

Note that since $\mathcal{U}$ is locally finite, it follows from (i) that the sum in (ii) is a finite sum, hence is well defined. A sheaf $\mathcal{S}$ is called fine if it has a partition of unity subordinate to any locally finite open covering of M .

For example, on any Riemann surface M , the sheaves $\mathcal{C}$ and $\mathcal{C}^{\infty}$ are fine sheaves. To see this, recall that for any locally finite open covering $\mathcal{U}$ of M there are C^{∞} functions r_{α} on M such that $r_{\alpha}(p) = 0$ for $p \in M - U_{\alpha}$ and $\Sigma_{\alpha} r_{\alpha}(p) = 1$; for a proof, see for instance L. Auslander and R. E. MacKenzie, Introduction to Differentiable Manifolds, (MacGraw-Hill, 1963). The operation of multiplication by the C^{∞} function r_{α} clearly defines a homomorphism η_{α} in the sheaf of germs of C^{∞} or of continuous functions, and these homomorphisms are a partition of unity for the sheaf.

Theorem 2. If $\mathcal{U} = \{U_{\alpha}\}$ is a locally finite open covering of a topological space M , and $\mathcal{S}$ is a fine sheaf on M , then $H^q(\mathcal{U}, \mathcal{S}) = 0$ for all $q > 0$. Hence, for any fine sheaf $\mathcal{S}$ over a paracompact Hausdorff space M, $H^q(M, \mathcal{S}) = 0$ for all $q > 0$.

Proof. Let $\{\eta_{\alpha}\}$ be a partition of unity for the sheaf $\mathcal{S}$ subordinate to the covering $\mathcal{U}$, and consider an arbitrary cocycle $f \in Z^q(\mathcal{U}, \mathcal{S})$, for $q > 0$. For any fixed index α and any (q-1) simplex $\sigma = (U_0, \ldots, U_{q-1})$, the induced homomorphism η^*_{α} on sections yields a section $\eta^*_{\alpha} f(U_{\alpha}, U_0, \ldots, U_{q-1}) \in \Gamma(U_{\alpha} \cap |\sigma|, \mathcal{S})$; since $\eta^*_{\alpha} f$ vanishes identically over $|\sigma| - U_{\alpha} \cap |\sigma|$, this section can be extended by values 0 to determine a section

$$g_{\alpha}(\sigma) = E_{|\sigma|} \eta^*_{\alpha} f(U_{\alpha}, U_0, \ldots, U_{q-1}) \in \Gamma(|\sigma|, \mathcal{S}) .$$

These sections then define a (q-1) cochain $g_{\alpha} \in C^{q-1}(\mathcal{U}, \mathcal{S})$. Note that, for a q-simplex $\sigma = (U_0, \ldots, U_q)$,

$$\delta g_\alpha(\sigma) = \sum_{i=0}^{q} (-1)^i \rho_{|\sigma|} g_\alpha(\sigma_i) \ , \quad \text{where} \quad \sigma_i = (U_0,\dots,U_{i-1},U_{i+1},\dots,U_q) \ ,$$

$$= \sum_{i=0}^{q} (-1)^i \rho_{|\sigma|} E_{|\sigma_i|} \eta_\alpha^* f(U_\alpha, U_0,\dots,U_{i-1} U_{i+1},\dots,U_q)$$

$$= \eta_\alpha^* f(\sigma) - E_{|\sigma|} \delta \eta_\alpha^* f = \eta_\alpha^* f(\sigma) \ ,$$

recalling that f is a cocycle. This holds for all indices α, and $g_\alpha(\sigma)$ vanishes identically outside the set U_α. The cochain $g = \Sigma_\alpha g_\alpha$ is well-defined, since $\mathcal{U}$ is locally finite, and $\delta g = \Sigma_\alpha \eta_\alpha^* f = f$, in view of the properties of a partition of unity. Thus any cocycle is cohomologous to zero, and $H^q(\mathcal{U}, \mathcal{S}) = 0$ as desired. This suffices to prove the theorem.

By applying this theorem, the cohomology groups can be described in the following frequently useful manner. A <u>fine resolution</u> of a sheaf $\mathcal{S}$ of abelian groups over a topological space M is an exact sequence of sheaves of abelian groups of the form

$$(3) \qquad 0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \cdots \ ,$$

where the sheaves $\mathcal{S}_i$ are all fine sheaves. For each of the sheaf homomorphisms d_i, there is the induced homomorphism of the groups of sections over an open subset $U \subset M$,

$$d_i^*: \Gamma(U, \mathcal{S}_i) \longrightarrow \Gamma(U, \mathcal{S}_{i+1}) \ ;$$

but the corresponding sequence of these groups and group homomorphisms is not generally exact.

<u>Theorem 3.</u> If (3) is a fine resolution of the sheaf $\mathcal{S}$ over a paracompact Hausdorff space M, then

$$H^q(M, \mathcal{S}) \cong (\text{kernel } d_q^*)/(\text{image } d_{q-1}^*) , \quad \text{for} \quad q > 0 .$$

Proof. Let $\mathcal{K}_i \subset \mathcal{S}_i$ be the kernel of the sheaf homomorphism d_i ; then the exact sequence (3) can be rewritten as the following collection of short exact sequences:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \xrightarrow{d_0} \mathcal{K}_1 \longrightarrow 0 ,$$

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{S}_i \xrightarrow{d_i} \mathcal{K}_{i+1} \longrightarrow 0 , \qquad i \geqq 1 .$$

A portion of the exact cohomology sequence associated to the first of the above short exact sequences is as follows:

$$\cdots \longrightarrow H^{q-1}(M, \mathcal{S}_0) \longrightarrow H^{q-1}(M, \mathcal{K}_1) \longrightarrow H^q(M, \mathcal{S}) \longrightarrow H^q(M, \mathcal{S}_0) \longrightarrow \cdots .$$

Since $\mathcal{S}_0$ is a fine sheaf and $q > 0$, it follows from Theorem 2 that $H^q(M, \mathcal{S}_0) = 0$. If $q = 1$, the formula of the present theorem follows immediately from this exact sequence since $H^{q-1}(M, \mathcal{K}_1) = H^0(M, \mathcal{K}_1) = (\text{kernel } d_1)$; while if $q > 1$, then $H^{q-1}(M, \mathcal{S}_0) = 0$ and $H^q(M, \mathcal{S}) \cong H^{q-1}(M, \mathcal{K}_1)$. The exact cohomology sequence associated to the second short exact sequence above, for the case $i = 1$, contains the terms

$$\cdots \to H^{q-2}(M, \mathcal{S}_1) \to H^{q-2}(M, \mathcal{K}_2) \to H^{q-1}(M, \mathcal{K}_1) \to H^{q-1}(M, \mathcal{S}_1) \to \cdots$$

Since $\mathcal{S}_1$ is fine and $q > 1$, $H^{q-1}(M, \mathcal{S}_1) = 0$. If $q = 2$, the desired result follows immediately, while if $q > 2$, then $H^{q-2}(M, \mathcal{S}_1) = 0$ as well, so $H^{q-2}(M, \mathcal{K}_2) \cong H^{q-1}(M, \mathcal{K}_1) \cong H^q(M, \mathcal{S})$. Continuing this process, the desired result follows eventually.

In many cases arising in practice, a fine resolution of a given sheaf appears naturally at hand, and the preceding theorem provides a very useful approach to the problem of calculating the cohomology groups; an illustration of this will be taken up next. But before turning to the example, it may be noted that a fine resolution can be constructed for any sheaf $\mathcal{S}$, so that the results of the preceding theorem can be applied theoretically to an arbitrary sheaf. For any sheaf $\mathcal{S}$, and any open subset $U \subset M$, let $\mathcal{S}_U^*$ be the set of all mappings $f: U \to \mathcal{S}$ such that $\pi \circ f: U \to U$ is the identity on U; for emphasis, note that the mappings f are not required to be continuous. The collection of these groups $\mathcal{S}_U^*$, together with the natural restriction mappings, form a complete presheaf over M; the associated sheaf $\mathcal{S}^*$ will be called the sheaf of discontinuous sections of $\mathcal{S}$. It is clear that $\mathcal{S}^*$ is a fine sheaf. For given any locally finite open covering $\mathcal{U} = \{U_\alpha\}$ of M, select some subsets $K_\alpha \subset U_\alpha$ so that $K_\alpha \cap K_\beta = \emptyset$ for $\alpha \neq \beta$ and $\bigcup_\alpha K_\alpha = M$; and define a mapping $\eta_\alpha: \mathcal{S}^* \to \mathcal{S}^*$ by putting $\eta_\alpha(s) = s$ if $s \in K_\alpha$, $\eta_\alpha(s) = 0$ if $s \notin K_\alpha$. It is easy to see that these maps η_α are sheaf homomorphisms, and they form a partition of unity for $\mathcal{S}^*$. Furthermore, there is a natural injection mapping $\mathcal{S} \to \mathcal{S}^*$. Now, to construct the fine resolution, put $\mathcal{S}_0 = \mathcal{S}^*$; put $\mathcal{S}_1 = (\mathcal{S}_0/\mathcal{S})^*$; and so on.

(e) For an example which will be of some use later in the present discussion, consider a connected open subset M of the complex line $\mathbb{C}$. Introduce on the space $\mathcal{C}_M^\infty$ the first-order linear partial differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) .$$

Note that the Cauchy-Riemann equations for a complex-valued function f can be written $\partial f/\partial \bar{z} = 0$; that is to say, given a function $f \in \mathcal{C}^{\infty}_{M}$, then $f \in \mathcal{O}_{M}$ if and only if $\partial f/\partial \bar{z} = 0$. The mapping $f \longrightarrow \partial f/\partial \bar{z}$ is a homomorphism from the ring $\mathcal{C}^{\infty}_{M}$ to itself; and hence this mapping induces a sheaf homomorphism

$$\bar{\partial}\colon \mathcal{C}^{\infty} \longrightarrow \mathcal{C}^{\infty} .$$

The Cauchy-Riemann condition can then be interpreted as the assertion that the kernel of this homomorphism is precisely the sheaf $\mathcal{O}$ of germs of holomorphic functions on M ; thus there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{C}^{\infty} \xrightarrow{\bar{\partial}} \mathcal{C}^{\infty} .$$

The sheaf $\mathcal{C}^{\infty}$ is a fine sheaf over M , as noted earlier; so this provides part of a fine resolution of the sheaf $\mathcal{O}$, and raises the question of whether this can be extended further as a fine resolution of $\mathcal{O}$.

<u>Lemma 6.</u> Let $g \in \mathcal{C}^{\infty}_{M}$, and let D be a connected open subset of the complex line $\mathbb{C}$ such that $\bar{D}$ is compact and $\bar{D} \subset M$. Then there exists a function $f \in \mathcal{C}^{\infty}_{M}$ such that $\partial f(z)/\partial \bar{z} = g(z)$ whenever $z \in D$.

Proof. Select a C^{∞} function r on the complex line $\mathbb{C}$ such that $r(z) = 1$ for $z \in \bar{D}$, $r(z) = 0$ for $z \in \mathbb{C} - M$, and $r(z) \neq 0$ only on a compact subset of $\mathbb{C}$. The function

$$h(z) = r(z)g(z) \quad \text{for} \quad z \in M, \qquad h(z) = 0 \quad \text{for} \quad z \in \mathbb{C} - M ,$$

is then a C^∞ function on the entire complex line, coincides with the given function g on the set $D \subset M$, and vanishes outside a compact subset of $\mathbb{C}$. Now put

$$f(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} h(z+\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta} ;$$

here the complex differential form notation is used, writing $d\zeta = d\xi + i d\eta$ if $\zeta = \xi + i\eta$, so that $d\zeta \wedge d\bar{\zeta} = -2i\, d\xi \wedge d\eta$, and thus $d\zeta \wedge d\bar{\zeta}$ is $(-2i)$ times the ordinary plane measure. Note that in terms of polar coordinates (r,θ), writing $\zeta = re^{i\theta}$, it follows that $(d\zeta \wedge d\bar{\zeta})/\zeta = (-2irdr \wedge d\theta)\cdot/re^{i\theta} = -2ie^{-i\theta}dr \wedge d\theta$, and hence $f(z)$ is clearly a well-defined and C^∞ function in the entire complex line $\mathbb{C}$. Differentiation yields the formula

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial h(z+\zeta)}{\partial \bar{z}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial h(z+\zeta)}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta}$$

$$= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{h(z+\zeta)}{\zeta}\right) d\zeta \wedge d\bar{\zeta} .$$

Now fix the point $z \in \mathbb{C}$. Select a disc Δ centered at the origin and large enough that the function $h(z+\zeta)$ vanishes identically for $\zeta \in \mathbb{C} - \Delta$; and a disc Δ_ϵ centered at the origin and of radius ϵ small enough that $\bar{\Delta}_\epsilon \subset \Delta$. The boundary of Δ will be called γ, and the boundary of Δ_ϵ will be called γ_ϵ ; here, γ and γ_ϵ are circles about the origin, with the positive orientation. Then

$$2\pi i \frac{\partial f}{\partial \bar{z}} = \lim_{\epsilon \to 0} \iint_{\Delta - \Delta_\epsilon} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{h(z+\zeta)}{\zeta}\right) d\zeta \wedge d\bar{\zeta} ;$$

and applying Stokes' theorem, and recalling that on the portion γ of the boundary of $\Delta - \Delta_\epsilon$ the integrand vanishes identically, secure that

$$2\pi i \frac{\partial f}{\partial \bar{z}} = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \frac{h(z+\zeta)}{\zeta} d\zeta .$$

Parametrize the circle γ_ϵ by writing $\zeta = \epsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$, so

$$2\pi i \frac{\partial f}{\partial \bar{z}} = \lim_{\epsilon \to 0} \int_{\theta=0}^{2\pi} h(z + \epsilon e^{i\theta}) i d\theta$$

$$= \int_{\theta=0}^{2\pi} h(z) i d\theta = 2\pi i \, h(z) .$$

Therefore $f(z)$ is the desired function, and the lemma is thereby proved.

As an immediate consequence of this lemma, if g is the germ of a C^∞ function at any point $p \in \mathbb{C}$, there exists a germ f of a C^∞ function at the point p such that $\partial f/\partial \bar{z} = g$. Consequently the following is an exact sequence of sheaves of abelian groups:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{\bar{\partial}} \mathcal{C}^\infty \longrightarrow 0 .$$

Considering the associated groups of sections, it follows immediately from Theorem 3 that

$$(4) \qquad \begin{cases} H^1(M, \mathcal{O}) \cong \Gamma(M, \mathcal{C}^\infty) \Big/ \frac{\partial}{\partial \bar{z}} \Gamma(M, \mathcal{C}^\infty), \text{ and} \\ H^q(M, \mathcal{O}) \cong 0 \qquad \text{for } q \geq 2 . \end{cases}$$

In fact, a slight extension of the above lemma leads to an interesting and useful result.

<u>Theorem 4.</u> Let M be a connected open subset of the complex line, and $g \in \mathcal{C}^\infty_M$. Then there exists a function $f \in \mathcal{C}^\infty_M$ such

that $\partial f(z)/\partial\bar{z} = g(z)$ for all $z \in M$.

Proof. Select a sequence of connected open subsets $D_n \subset M$ with the following properties:

(i) $\bar{D}_n$ is compact, and $\bar{D}_n \subset D_{n+1}$;

(ii) $\bigcup_{n=1}^{\infty} D_n = M$;

(iii) any function holomorphic in D_{n-1} can be approximated uniformly well on $\bar{D}_{n-2} \subset D_{n-1}$ by functions holomorphic on D_n .

(The last condition is an approximation theorem of the Runge sort; to see that this construction is possible, see for instance E. Hille, Analytic Function Theory, vol. II, pp. 299 ff., (Ginn and Co., 1962).)

Next, by induction on the index n , observe that there is a sequence of functions f_n with the following properties:

(iv) f_n is a C^∞ function in D_n ;

(v) $\partial f_n(z)/\partial\bar{z} = g$ for all $z \in D_n$;

(vi) $|f_n(z) - f_{n-1}(z)| < 2^{-n}$ for all $z \in \bar{D}_{n-2}$.

To see this, suppose given the functions $f_1, \ldots, f_{n-1}$, for some index $n \geqq 0$. By Lemma 6, there is a function $h_n \in \mathcal{E}_M^\infty$ such that $\partial h_n(z)/\partial\bar{z} = g(z)$ whenever $z \in D_n$. In case $n = 0$ or 1 , there is nothing further to show. In case $n \geqq 2$, the functions h_n and f_{n-1} are both C^∞ in D_{n-1} , and $\partial(h_n(z) - f_{n-1}(z)/\partial\bar{z} = g(z) - g(z) = 0$ for $z \in D_{n-1}$; that is, $h_n(z) - f_{n-1}(z)$ is holomorphic in D_{n-1} . There exists a function $h(z)$ holomorphic in D_n such that $|h_n(z) - f_{n-1}(z) - h(z)| < 2^{-n}$ for all $z \in \bar{D}_{n-2}$, as a consequence of the approximation property (iii) above. The function $f_n(z) = h_n(z) - h(z)$ then satisfies the desired conditions.

Now for any point $z \in M$, the sequence $\{f_n(z)\}$ converges to some limiting value $f(z)$. Indeed, for all points $z \in D_n$,

$$f(z) = f_{n+2}(z) + \sum_{m=n+2}^{\infty} (f_{m+1}(z) - f_m(z)) .$$

Since $|f_{m+1}(z) - f_m(z)| < 2^{-m}$ for $z \in D_n \subset D_{m-2}$, $m \geq n+2$, by (vi), the series is absolutely uniformly convergent in D_n; and since the individual terms of the series are holomorphic in D_n by (v), the sum is also holomorphic. Therefore $f(z)$ is C^∞ in D_n; and $\partial f(z)/\partial \bar{z} = \partial f_{n+2}(z)/\partial \bar{z} = g(z)$ in D_n by (v). This suffices to conclude the proof.

Corollary. If M is a connected open subset of the complex line $\mathbf{C}$, then

$$H^q(M, \mathcal{O}) = 0 \quad \text{for} \quad q \geq 1 .$$

Proof. This result is an immediate consequence of formula (4) and the preceding theorem.

(f) The cohomology groups of a space with coefficients in a sheaf have been defined as direct limits of cohomology groups of coverings of that space. It is natural to ask when the cohomology of the space can be read directly from the cohomology of some covering; and the answer is provided by the following result.

Theorem 5. Let $\mathcal{S}$ be a sheaf of abelian groups over a paracompact Hausdorff space M, and $\mathcal{U} = \{U_\alpha\}$ be an open covering of M such that

$$H^q(|\sigma|, \mathcal{S}) = 0 \quad \text{for all} \quad \sigma \in N(\mathcal{U}) \quad \text{and} \quad q \geq 1 .$$

Then

$$H^q(M, \mathcal{S}) \cong H^q(\mathfrak{U}, \mathcal{S}) \quad \text{for all} \quad q \geqq 0 .$$

Proof. Select a fine resolution

$$(5) \qquad 0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \cdots$$

of the sheaf $\mathcal{S}$ over M. Then for the induced homomorphisms $d_i^*: \Gamma(M, \mathcal{S}_i) \longrightarrow \Gamma(M, \mathcal{S}_{i+1})$, it follows from Theorem 3 that $H^q(M, \mathcal{S}) \cong (\text{kernel } d_q^*)/(\text{image } d_{q-1}^*)$ for all $q \geq 1$. For any simplex $\sigma \in N(\mathfrak{U})$, the cohomology groups $H^q(|\sigma|, \mathcal{S})$ are determined similarly by restricting the resolution (5) to $|\sigma|$; but since $H^q(|\sigma|, \mathcal{S}) = 0$ by hypothesis, it follows that the sequence of sections

$$0 \longrightarrow \Gamma(|\sigma|, \mathcal{S}) \longrightarrow \Gamma(|\sigma|, \mathcal{S}_0) \xrightarrow{d_0^*} \Gamma(|\sigma|, \mathcal{S}_1) \xrightarrow{d_1^*} \cdots$$

corresponding to (5) is actually an exact sequence. Since further the cochain groups are merely direct sums of groups of sections over the various simplices of $N(\mathfrak{U})$, there follows an exact sequence of groups of the form

$$(6) \quad 0 \longrightarrow C^q(\mathfrak{U}, \mathcal{S}) \longrightarrow C^q(\mathfrak{U}, \mathcal{S}_0) \xrightarrow{d_0^*} C^q(\mathfrak{U}, \mathcal{S}_1) \xrightarrow{d_1^*} \cdots .$$

The coboundary maps commute with the homomorphisms of the exact sequence (6), so that all of these sequences can be group together in the following commutative diagram:

$$
\begin{array}{ccccccccc}
(7) & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(M, \mathcal{S}) & \longrightarrow & \Gamma(M, \mathcal{S}_0) & \xrightarrow{d_0^*} & \Gamma(M, \mathcal{S}_1) & \xrightarrow{d_1^*} & \Gamma(M, \mathcal{S}_2) \to \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & C^0(\mathfrak{U}, \mathcal{S}) & \to & C^0(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{d_0^*} & C^0(\mathfrak{U}, \mathcal{S}_1) & \xrightarrow{d_1^*} & C^0(\mathfrak{U}, \mathcal{S}_2) \to \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
0 & \to & C^1(\mathfrak{U}, \mathcal{S}) & \to & C^1(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{d_0^*} & C^1(\mathfrak{U}, \mathcal{S}_1) & \xrightarrow{d_1^*} & C^1(\mathfrak{U}, \mathcal{S}_2) \to \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
0 & \to & C^2(\mathfrak{U}, \mathcal{S}) & \to & C^2(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{d_0^*} & C^2(\mathfrak{U}, \mathcal{S}_1) & \xrightarrow{d_1^*} & C^2(\mathfrak{U}, \mathcal{S}_2) \to \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
$$

All of the rows except for the first are exact, from the exactness of (6); and the measure of inexactness of the first row, in the obvious sense, is the cohomology of M. Since all the sheaves $\mathcal{S}_i$ are fine, all of the columns except the first are also exact sequences, by Theorem 2; and the measure of inexactness of the first column is the cohomology of the covering $\mathfrak{U}$. The desired result follows immediately from a diagram chase through (7); the details will be left to the reader.

As a terminological convenience, a covering $\mathfrak{U}$ of the space M which satisfies the conditions of Theorem 5 will be called a Leray covering of M for the sheaf $\mathcal{S}$.

Corollary 1. If $\mathfrak{U}$ and $\mathfrak{V}$ are Leray coverings of a paracompact Hausdorff space M for a sheaf $\mathcal{S}$, and $\mu: \mathfrak{V} \to \mathfrak{U}$ is a refinement, then the induced mapping

$$\mu^*: H^q(\mathfrak{U}, \mathcal{S}) \to H^q(\mathfrak{V}, \mathcal{S})$$

is an isomorphism.

Proof. The natural homomorphisms $u: H^q(\mathcal{U}, \mathcal{S}) \longrightarrow H^q(M, \mathcal{S})$ and $v: H^q(\mathcal{V}, \mathcal{S}) \longrightarrow H^q(M, \mathcal{S})$ are isomorphisms, by Theorem 5; and since $v \circ \mu^* = u$, it follows that necessarily μ^* is an isomorphism.

<u>Corollary 2.</u> If $\mathcal{U}$ is an arbitrary open covering of a paracompact Hausdorff space M, the natural mapping

$$u: H^1(\mathcal{U}, \mathcal{S}) \longrightarrow H^1(M, \mathcal{S})$$

is an injection, (i.e., has kernel 0).

Proof. For an arbitrary open covering $\mathcal{U}$ of M there corresponds a commutative diagram (7); the columns are all exact, except for the first column, but without the hypothesis that $\mathcal{U}$ is a Leray covering, the rows need not be exact beyond the second place. However, a diagram chase shows that it is still possible to conclude that the mapping $H^1(\mathcal{U}, \mathcal{S}) \longrightarrow H^1(M, \mathcal{S})$ is an injection; details again will be left to the reader.

§4. Divisors and line bundles

(a) One of the main approaches to function theory on Riemann surfaces involves the study of functions from properties of their zeros and singularities. The sheaf machinery developed in the preceding two sections proves quite useful here. On a fixed Riemann surface M, consider the sheaves $\mathcal{O}^*$ of germs of nowhere-vanishing holomorphic functions and $\mathcal{M}^*$ of germs of not identically vanishing meromorphic functions; in both cases the group structure in the sheaf is multiplicative, and $\mathcal{O}^* \subset \mathcal{M}^*$. The quotient sheaf $\mathcal{D} = \mathcal{M}^*/\mathcal{O}^*$ is called the sheaf of germs of divisors on the Riemann surface. A section of the sheaf $\mathcal{D}$ over a subset $U \subset M$ will be called a divisor on the subset U. Note that a germ of a divisor at a point $p \in M$, that is, an element of the stalk $\mathcal{D}_p$, is an equivalence class of meromorphic functions, where two meromorphic functions are considered as equivalent when their quotient is holomorphic and nowhere vanishing; thus an equivalence class consists of all the germs of meromorphic functions having the same order (the same zero or pole) at the point p. In this sense, divisors merely furnish a description of the zeros and singularities of meromorphic functions.

In the case of a single complex variable, the sheaf $\mathcal{D}$ has an alternative and much simpler description; and this simplicity is one of the distinctive differences between the function theory of one and of several complex variables. For any germ $f \in \mathcal{M}_p^*$, the equivalence class of f in $\mathcal{D}_p$ is described uniquely by the order $v_p(f)$ of the function f at the point p; the stalk $\mathcal{D}_p = \mathcal{M}_p^*/\mathcal{O}_p^*$

is therefore naturally isomorphic to the additive group of the integers. (Recall that $\nu_p(f\cdot g) = \nu_p(f) + \nu_p(g)$, so that the multiplicative structure in $\mathcal{M}_p^*$ corresponds to the additive structure of the orders $\nu_p(f) \in \mathbb{Z}$.) To describe the topology of $\mathcal{D} = \mathcal{M}^*/\mathcal{O}^*$, recall that such a quotient sheaf is always topologized by defining the images of sections of $\mathcal{M}^*$ over a basis of the open sets of M, as a basis for the open sets of $\mathcal{D}$. Now for any open subset U and any meromorphic function $f \in \Gamma(U, \mathcal{M}^*)$, the image set in $\mathcal{D}$ is the divisor of the function f; and the important thing to note is that the order of a meromorphic function f is non-zero only at a discrete set of points in U. Thus an open set in $\mathcal{D}$ will consist of an integer associated to the points of an open subset $U \subset M$, in such a manner that non-zero integers appear only for a discrete set of points in U. It is thus clear that the sheaf $\mathcal{D}$ can be described as follows. To any open subset $U_\alpha \subset M$ associate the additive group $\mathcal{D}_\alpha$ of all mappings $\nu: U_\alpha \to \mathbb{Z}$ such that $\nu(p) \neq 0$ only on a discrete subset of U_α; the group structure is of course the pointwise addition of the functions. If $U_\beta \subset U_\alpha$, the natural restriction of such functions from U_α to U_β is a group homomorphism $\rho_{\beta\alpha}: \mathcal{D}_\alpha \to \mathcal{D}_\beta$. This defines a complete presheaf over M, and the associated sheaf is just the sheaf $\mathcal{D}$ of germs of divisors. This latter description will generally be used henceforth. Note that, from this description, it is obvious that $\mathcal{D}$ <u>is a fine sheaf over</u> M; the details of the verification will be left to the reader.

As for notation, divisors will generally be denoted by German script d, namely, $\mathfrak{d}$. To describe a divisor $\mathfrak{d} \in \Gamma(U, \mathcal{D})$, it is of course sufficient to give the orders (the integers) at only those points of the discrete subset of U where the order is non-zero; thus divisors will be written

$\mathcal{J} = \Sigma_i \nu_i \cdot p_i$, where $\nu_i \in \mathbf{Z}$, $p_i \in U$, and $\cup_i p_i \subset U$ is discrete. For a meromorphic function $f \in \Gamma(U, \mathcal{M}^*)$, the divisor of f will be denoted by $\mathcal{J}(f)$; thus

$$\mathcal{J}(f) = \Sigma_{p \in U} \nu_p(f) \cdot p ,$$

where the sum can be restricted to the discrete subset of U consisting of points at which $\nu_p(f) \neq 0$. Note that $\mathcal{J}(fg) = \mathcal{J}(f) + \mathcal{J}(g)$; and that $\mathcal{J}(f)$ is not defined for the function $f \equiv 0$. The divisors over U can be given a partial ordering by defining

$$\mathcal{J} = \Sigma_i \nu_i \cdot p_i \geqq 0 \quad \text{provided} \quad \nu_i \geqq 0 .$$

Note then that holomorphic functions f over U are characterized by the condition that $\mathcal{J}(f) \geqq 0$; and more generally, $\mathcal{J}(f) \geqq \mathcal{J}(g)$ if and only if f/g is holomorphic. Divisors $\mathcal{J}$ such that $\mathcal{J} \geqq 0$ will be called <u>positive divisors</u>.

The mapping which associates to a meromorphic function f its divisor $\mathcal{J}(f)$ is just the natural homomorphism $\mathcal{J} : \mathcal{M}^* \longrightarrow \mathcal{D}$ from the sheaf $\mathcal{M}^*$ to its quotient sheaf; this can be described by writing the exact sequence of sheaves

(1) $$0 \longrightarrow \mathcal{O}^* \xrightarrow{i} \mathcal{M}^* \xrightarrow{\mathcal{J}} \mathcal{D} \longrightarrow 0 ,$$

where i is the natural inclusion mapping. (The notation 0 will always be used for the trivial sheaf, whether the group structure of the stalk is considered additive or multiplicative.) Corresponding to this sheaf sequence over M is the familiar exact cohomology sequence, in which appears the homomorphism $\mathcal{D}^*: \Gamma(M, \mathcal{M}^*) \longrightarrow \Gamma(M, \mathcal{D})$. An element $\mathcal{D} \in \Gamma(M, \mathcal{D})$ is a divisor defined over the entire Riemann surface M ; while an element in the image of $\mathcal{D}^*$ is the divisor of a meromorphic function defined over all of M. That there exist non-trivial divisors defined over all of M, or equivalently that $\Gamma(M, \mathcal{D}) \neq 0$, is completely obvious; but that there exist non-trivial meromorphic functions defined over all of M, or equivalently that $\Gamma(M, \mathcal{M}) \neq \mathbb{C}$, is far from trivial, is indeed one of the basic existence theorems of the subject. Thus the question of whether or not the mapping $\mathcal{D}^*$ is onto is one of some interest. In a special case, the answer is immediate.

Theorem 6. (Weierstrass' theorem) If M is any connected open subset of the complex line $\mathbb{C}$, the following is an exact sequence of groups:

$$0 \longrightarrow \Gamma(M, \mathcal{O}^*) \xrightarrow{i^*} \Gamma(M, \mathcal{M}^*) \xrightarrow{\mathcal{D}^*} \Gamma(M, \mathcal{D}) \longrightarrow 0 .$$

Proof. The exact cohomology sequence corresponding to the exact sheaf sequence (1) begins as follows:

$$0 \longrightarrow \Gamma(M, \mathcal{O}^*) \longrightarrow \Gamma(M, \mathcal{M}^*) \longrightarrow \Gamma(M, \mathcal{D}) \longrightarrow H^1(M, \mathcal{O}^*) \longrightarrow \dots ;$$

therefore to prove the theorem, it suffices to show that $H^1(M, \mathcal{O}^*) = 0$. Recalling the exact sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0, \quad \text{when} \quad e(f) = \exp. 2\pi i f ,$$

there is an associated cohomology sequence, which includes the segment

$$H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \longrightarrow H^2(M, \mathbf{Z}) \longrightarrow H^2(M, \mathcal{O}) .$$

Now by the collorary to Theorem 4, $H^1(M, \mathcal{O}) = H^2(M, \mathcal{O}) = 0$; so that $H^1(M, \mathcal{O}^*) \cong H^2(M, \mathbf{Z})$. But since M is a non-compact two-dimensional manifold, $H^2(M, \mathbf{Z}) = 0$, and therefore $H^1(M, \mathcal{O}^*) = 0$, as desired. (See the topological appendix for a discussion of the assertion that $H^2(M, \mathbf{Z}) = 0$.)

Remarks. The corresponding theorem holds for an arbitrary non-compact Riemann surface M ; the only result needed is that $H^1(M, \mathcal{O}) = H^2(M, \mathcal{O}) = 0$. (For the proof, see for instance R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, p. 270, (Prentice-Hall, 1965).) The theorem implies that an arbitrary divisor on M is the divisor of a global meromorphic function on M . The Weierstrass factor-theorem gives an explicit representation for a function with the prescribed divisor; (see for instance L. Ahlfors, Complex Analysis, p. 157, (McGraw-Hill, 1953)).

For compact Riemann surfaces the preceding theorem does not hold at all; we shall see eventually that $H^1(M, \mathcal{O}^*) \neq 0$. An investigation of the precise extent to which the theorem fails will be one of the main topics of consideration. A few trivial observations and further definitions are in place here. Recall that the sheaf $\mathcal{D}$ of germs of divisors on a Riemann surface was noticed above to be a fine sheaf; hence by Theorem 2, $H^1(M, \mathcal{D}) = 0$. Therefore the exact cohomology sequence associated to the exact sheaf sequence (1) has the form

$$(2) \qquad 0 \longrightarrow \Gamma(M, \mathcal{O}^*) \xrightarrow{i^*} \Gamma(M, \mathcal{M}^*) \xrightarrow{\mathcal{D}^*} \Gamma(M, \mathcal{D}) \xrightarrow{\delta^*} H^1(M, \mathcal{O}^*) \xrightarrow{i^*} H^1(M, \mathcal{M}^*) \longrightarrow 0 .$$

The quotient group

$$\underset{\sim}{A}(M) = \Gamma(M, \mathcal{D})/ \mathcal{D}^*\Gamma(M, \mathcal{M}^*)$$

is really the measure of the extent to which Theorem 6 fails to hold; and in terms of that group the exact sequence (2) can be written

$$(3) \qquad 0 \longrightarrow \underset{\sim}{A}(M) \longrightarrow H^1(M, \mathcal{O}^*) \longrightarrow H^1(M, \mathcal{M}^*) \longrightarrow 0$$

As a matter of terminology, the group $\Gamma(M, \mathcal{D})$ will be called the <u>group of divisors</u> on M. Two divisors $\mathcal{D}_1, \mathcal{D}_2 \in \Gamma(M, \mathcal{D})$ will be called <u>linearly equivalent</u>, written $\mathcal{D}_1 \approx \mathcal{D}_2$, if their difference is the divisor of a meromorphic function on M, that is, if $\mathcal{D}_1 - \mathcal{D}_2 = \mathcal{D}(f)$ for some $f \in \Gamma(M, \mathcal{M}^*)$. This is an equivalence relation, is indeed the equivalence relation corresponding to the homomorphism $\mathcal{D}^*$ in (2); in particular, the image of $\mathcal{D}^*$ is the group of divisors linearly equivalent to zero. The group $\underset{\sim}{A}(M)$ is called the <u>divisor class group</u> on M, and is the group of linear equivalence classes of divisors on M. The exact sequence (3) will later permit a rather complete description of the group $\underset{\sim}{A}(M)$, and thus settle the question of the extent to which the Weierstrass theorem holds on compact Riemann surfaces.

(b) In the further discussion of these questions, one is led in a very natural manner to investigate a special class of sheaves. To introduce these sheaves, consider the group $H^1(M, \mathcal{O}^*)$, which appeared notably in the above discussion. This group will be called the <u>group of complex line bundles over M</u>; and a cohomology class $\xi \in H^1(M, \mathcal{O}^*)$ will be called a <u>complex line bundle</u> over M.

(The terminology arises from an interesting geometric interpretation which can be given to the elements $\xi \in H^1(M, \mathcal{O}^*)$; this geometric interpretation is totally irrelevant to the purposes at hand, although it is not uncommonly injected into the discussion of these topics, and will be ignored here. The interested reader is referred to F. Hirzebruch, Neue Topologische Methoden in der Algebraischen Geometrie, (Springer, 1956).)

For any complex line bundle $\xi \in H^1(M, \mathcal{O}^*)$, select a basis $\mathcal{U} = \{U_\alpha\}$ for the open sets of M , and a cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ representing that cohomology class; since bases are cofinal in the open coverings of M , there always exists such a representation. The elements $\xi_{\alpha\beta}$ are holomorphic, nowhere-vanishing functions defined in the open sets $U_\alpha \cap U_\beta$, and the cocycle condition asserts that $\xi_{\alpha\beta}(p) \cdot \xi_{\beta\gamma}(p) = \xi_{\alpha\gamma}(p)$ whenever $p \in U_\alpha \cap U_\beta \cap U_\gamma$. To each open set $U_\alpha \in \mathcal{U}$ associate the group $\mathcal{S}_\alpha = \Gamma(U_\alpha, \mathcal{O})$ of holomorphic functions in U_α . To each inclusion relation $U_\beta \subset U_\alpha$ associate the group homomorphism $\rho_{\beta\alpha}: \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\beta$, which associates to a function $f \in \mathcal{S}_\alpha = \Gamma(U_\alpha, \mathcal{O})$ the function $\rho_{\beta\alpha}(f) \in \mathcal{S}_\beta = \Gamma(U_\beta, \mathcal{O})$ defined by

$$(\rho_{\beta\alpha}f)(p) = \xi_{\beta\alpha}(p) \cdot f(p) \quad \text{for} \quad p \in U_\beta \subset U_\alpha .$$

Note that whenever $U_\gamma \subset U_\beta \subset U_\alpha$ and $f \in \mathcal{S}_\alpha$, then $(\rho_{\gamma\beta}(\rho_{\beta\alpha}f))(p) = \xi_{\gamma\beta}(p) \cdot \xi_{\beta\alpha}(p) \cdot f(p) = \xi_{\gamma\alpha}(p) \cdot f(p) = (\rho_{\gamma\alpha}f)(p)$ for all $p \in U_\gamma$; that is, $\rho_{\gamma\beta}\rho_{\beta\alpha} = \rho_{\gamma\alpha}$. Therefore $\{\mathcal{U}, \mathcal{S}_\alpha, \rho_{\alpha\beta}\}$ is a presheaf over M , which is readily seen to be a complete presheaf; the associated sheaf is called the <u>sheaf of germs of holomorphic cross-sections of the line bundle</u> ξ , and will be denoted by $\mathcal{O}(\xi)$. It is a straightforward exercise, which will be left to the reader, to show that the sheaf $\mathcal{O}(\xi)$ is defined independently of the choice of cocycle representing the cohomology class ξ, that is, that the sheaves constructed in terms of two cocycles representing the same cohomology class are isomorphic sheaves.

Since the above presheaf is complete, there is a natural identification $\Gamma(U_\alpha, \mathcal{O}(\xi)) = \mathcal{S}_\alpha \cong \Gamma(U_\alpha, \mathcal{O})$. It is then clear that an element $f \in \Gamma(M, \mathcal{O}(\xi))$ corresponds to a collection $\{f_\alpha\}$, where $f_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ and

(4) $$f_\alpha(p) = \xi_{\alpha\beta}(p) \cdot f_\beta(p) \quad \text{whenever} \quad p \in U_\alpha \cap U_\beta ;$$

these sections of $\mathcal{O}(\xi)$ will also be called <u>holomorphic cross-sections of the line bundle</u> ξ. Note that the set of all such sections has the structure of a complex vector space, as well as just that of an abelian group; and that $\mathcal{O} = \mathcal{O}(1)$, where $1 \in H^1(M, \mathcal{O}^*)$ is the trivial line bundle.

The construction just described could have been carried through just as well for the groups $\mathcal{S}'_\alpha = \Gamma(U_\alpha, \mathcal{M})$; the homomorphisms $\rho_{\beta\alpha}$ are well defined, as above, and the collection $\{\mathfrak{U}; \mathcal{S}'_\alpha, \rho_{\alpha\beta}\}$ is again a complete presheaf. The associated sheaf will be called the <u>sheaf of germs of meromorphic cross-sections of the line bundle</u> ξ, and will be denoted by $\mathcal{M}(\xi)$. The elements $f \in \Gamma(M, \mathcal{M}(\xi))$ correspond to collections $\{f_\alpha\}$, when the functions f_α are now meromorphic functions satisfying the relations (4); such sections will also be called <u>meromorphic cross-sections of the line bundle</u> ξ. Or, in the same manner, using the groups $\mathcal{S}''_\alpha = \Gamma(U_\alpha, \mathcal{C}^\infty)$, there arises a sheaf $\mathcal{C}^\infty(\xi)$ which will be called the <u>sheaf of germs of</u> C^∞ <u>cross-sections of the line bundle</u> ξ. The sheaf $\mathcal{C}^\infty(\xi)$ is of course always a fine sheaf, an observation which will be of use later.

For a cross-section $f \in \Gamma(M, \mathcal{M}^*(\xi))$, the <u>order</u> of f at a point p is a well-defined integer $\nu_p(f)$; for defining $\nu_p(f) = \nu_p(f_\alpha)$ when $p \in U_\alpha$, and recalling that the meromorphic functions $\{f_\alpha\}$ satisfy equations (4) where $\xi_{\alpha\beta}$ are holomorphic nowhere-vanishing functions, it follows that $\nu_p(f_\alpha) = \nu_p(f_\beta)$ whenever $p \in U_\alpha \cap U_\beta$. Note that for any section f which is not identically zero, the order is non-zero only on a discrete set of points; hence to the section f there is associated a well-defined divisor

$$\mathcal{S}(f) = \Sigma_{p \in M}\, \nu_p(f)\cdot p ,$$

called the <u>divisor of the cross-section</u> $f \in \Gamma(M, \mathcal{M}(\xi))$. Then $\Gamma(M, \mathcal{O}(\xi)) \subset \Gamma(M, \mathcal{M}(\xi))$ appears merely as the subgroup of meromorphic cross-sections of the line bundle ξ having positive divisor, that is,

$$\Gamma(M, \mathcal{O}(\xi)) = \{f \in \Gamma(M, \mathcal{M}(\xi)) \mid \mathcal{S}(f) \geqq 0\} .$$

One further general remark of importance is that, for any line bundle $\xi \in H^1(M, \mathcal{O}^*)$, $\xi = 1$ (the trival line bundle) if and only if there exists a cross-section $f \in \Gamma(M, \mathcal{O}(\xi))$ such that $\mathcal{S}(f) = 0$. For $\mathcal{S}(f) = 0$ means that the functions $\{f_\alpha\}$ are holomorphic and nowhere-vanishing in U_α, and from equation (4) they form a zero-cochain in $C^0(\mathcal{U}, \mathcal{O}^*)$ having ξ as its coboundary, so that $\xi = 1$; conversely, if $\xi = 1$, then $\Gamma(M, \mathcal{O}(\xi)) = \Gamma(M, \mathcal{O})$, and this contains the non-zero constant functions. Of course, in a parallel manner, for a cohomology class $\xi \in H^1(M, \mathcal{M}^*)$, $\xi = 1$ if and only if there is a cross-section $f \in \Gamma(M, \mathcal{M}(\xi))$ which is not identically zero. Recalling the exact sequence (3), every element $\xi \in H^1(M, \mathcal{M}^*)$ can be represented by an element of $H^1(M, \mathcal{O}^*)$; therefore one can assert that on any Riemann surface, <u>$H^1(M, \mathcal{M}^*) = 0$ if and only if, for every line bundle $\xi \in H^1(M, \mathcal{O}^*)$, $\Gamma(M, \mathcal{M}(\xi)) \neq 0$</u>. The vanishing of the cohomology group $H^1(M, \mathcal{M}^*)$ is therefore equivalent to the fundamental existence theorem for Riemann surfaces, namely, the theorem that every line bundle has a non-trivial (not identically vanishing) meromorphic cross-section; and this is also of course equivalent to the assertion that every line bundle is the line bundle of a divisor on the surface.

To any divisor $\mathcal{J} \in \Gamma(M, \mathcal{D})$ there is associated by the exact sequence (2) a line bundle $\delta^* \mathcal{J} \in H^1(M, \mathcal{O}^*)$, and hence also the sheaf $\mathcal{O}(\delta^* \mathcal{J})$ of germs of holomorphic cross-sections of that line bundle; to simplify the notation, set $\mathcal{O}(\mathcal{J}) = \mathcal{O}(\delta^* \mathcal{J})$. Now the sheaf $\mathcal{O}(\mathcal{J})$ has another interpretation of interest as well. To the divisor $\mathcal{J}$ associate a subsheaf $\mathcal{O}_{\mathcal{M}}(\mathcal{J}) \subset \mathcal{M}$ defined as follows. For any point $p \in M$ let

$$\mathcal{O}_{\mathcal{M}}(\mathcal{J})_p = \{f \in \mathcal{M}_p \mid \text{either } f \equiv 0 \text{ or } \mathcal{J}(f) \geqq \mathcal{J} \text{ near } p\};$$

and put $\mathcal{O}_{\mathcal{M}}(\mathcal{J}) = \cup_{p \in M} \mathcal{O}_{\mathcal{M}}(\mathcal{J})_p$. It is clear that each $\mathcal{O}_{\mathcal{M}}(\mathcal{J})_p \subset \mathcal{M}_p$ is a subgroup, and that $\mathcal{O}_{\mathcal{M}}(\mathcal{J}) \subset \mathcal{M}$ is an open subset; hence $\mathcal{O}_{\mathcal{M}}(\mathcal{J})$ is a well-defined subsheaf of $\mathcal{M}$.

<u>Lemma 7.</u> The sheaves $\mathcal{O}(\mathcal{J})$ and $\mathcal{O}_{\mathcal{M}}(\mathcal{J})$ are canonically isomorphic.

Proof. It is necessary to examine the homomorphism δ^* in the exact sequence (2) a bit more closely. For the given divisor $\mathcal{J} \in \Gamma(M, \mathcal{D})$, it follows from the exactness of the sheaf sequence (1) that there are open sets $\{U_\alpha\}$ forming a covering $\mathcal{U}$ of M, and meromorphic functions d_α defined in the various sets U_α, such that $\mathcal{J}(d_\alpha) = \mathcal{J}|U_\alpha$. Then in each intersection $U_\alpha \cap U_\beta$ the function $\xi_{\alpha\beta} = d_\beta/d_\alpha$ is holomorphic and nowhere-vanishing; and the collection $(\xi_{\alpha\beta})$ of all such functions define the line bundle $\xi = \delta^*\mathcal{J} \in H^1(M, \mathcal{O}^*)$. (The functions (d_α) form a zero-cochain in $C^0(\mathcal{U}, \mathcal{M}^*)$ which maps onto the zero-cocycle $\mathcal{J} \in Z^0(\mathcal{U}, \mathcal{D})$; and the functions $(\xi_{\alpha\beta})$ form the one-cocycle which is the coboundary of the cochain (d_α); recall the proof of Theorem 1.) To each germ $f \in \mathcal{O}_M(\mathcal{J})_p \subset \mathcal{M}_p$ and to each open set U_α containing p associate the germ $f_\alpha = f/d_\alpha \in \mathcal{M}_p$. Since $\mathcal{J}(f_\alpha) = \mathcal{J}(f) - \mathcal{J}(d_\alpha) = \mathcal{J}(f) - \mathcal{J} \geqq 0$ near p, the germ f_α will necessarily be holomorphic at p; and if $p \in U_\alpha \cap U_\beta$, then $f_\alpha = f/d_\alpha = f \cdot \xi_{\alpha\beta}/d_\beta = \xi_{\alpha\beta} \cdot f_\beta$. Therefore the functions (f_α) define the germ of an element in $\mathcal{O}(\xi)_p = \mathcal{O}(\mathcal{J})_p$. This defines a mapping from $\mathcal{O}_M(\mathcal{J})$ to $\mathcal{O}(\mathcal{J})$, which is readily seen to be an isomorphism, and thus completes the proof.

Since the sheaf $\mathcal{O}(\mathcal{J})$ is defined in terms of holomorphic functions, it is the easier to handle analytically and will play the greater role in the present discussions. However, in view of the isomorphism $\mathcal{O}(\mathcal{J}) \cong \mathcal{O}_M(\mathcal{J})$, any results about the sheaf $\mathcal{O}(\mathcal{J})$ carry over to results about the sheaf $\mathcal{O}_M(\mathcal{J})$; and this re-interpretation frequently leads to interesting statements. For example,

$\Gamma(M, \mathcal{O}_M(\mathfrak{d}))$ is the vector space consisting of those meromorphic functions f defined on the entire Riemann surface M such that $\mathfrak{d}(f) \geqq \mathfrak{d}$; and the dimension of this vector space is a number of some interest.

(c) For a compact Riemann surface M the spaces $H^q(M, \mathcal{O}(\xi))$ are finite-dimensional complex vector spaces for all dimensions $q \geq 0$ and any line bundle $\xi \in H^1(M, \mathcal{O}^*)$; in fact $H^q(M, \mathcal{O}(\xi)) = 0$ for $q \geq 2$, as we shall see in the following section, so it suffices to prove the finite-dimensionality only for $q = 0$ and 1 . In demonstrating this, it is convenient to topologize the cochain and cohomology groups and apply a few simple results of the theory of topological vector spaces. At this stage there is a choice to make, since one can either topologize the full spaces of cochains (as Frechet spaces), or pass to certain subspaces of cochains which admit simpler structures as topological vector spaces (namely, as Hilbert spaces). The latter approach has been selected here, to minimize prerequisites; but the arguments are basically the same in either approach, and in higher dimensions as well, following H. Cartan and J. P. Serre, (C. R. Acad. Sci. Paris, 237(1953), 128-130).

First, let $U \subset \mathbf{C}$ be a connected open subset of the complex line, with $z = x + iy$ as the complex coordinate function on U , so that $dx \wedge dy$ is the standard Euclidean plane measure in U. Define

$$\Gamma_0(U, \mathcal{O}) = \{f \in \Gamma(U, \mathcal{O}) \mid \iint_U |f(z)|^2 dx \wedge dy < \infty\} ;$$

thus $\Gamma_0(U, \mathcal{O}) \subset \Gamma(U, \mathcal{O})$ is a vector subspace, which will be called the space of <u>square-integrable analytic functions</u> in U. For any two functions $f, g \in \Gamma_0(U, \mathcal{O})$, it follows immediately from the Cauchy-Schwarz inequality that

$$(5) \qquad (f,g)_U = \iint_U f(z)\overline{g(z)}dx \wedge dy$$

is a well-defined positive definite Hermitian inner product on the space $\Gamma_0(U, \mathcal{O})$, in terms of which $\Gamma_0(U, \mathcal{O})$ is a pre-Hilbert space, (that is, $\Gamma_0(U, \mathcal{O})$ is a Hilbert space in all but completeness); the norm in this space is given by $\|f\|_U^2 = (f,f)_U$. Now if $f \in \Gamma_0(U, \mathcal{O})$, $z_0 \in U$, and $\Delta(z_0,r)$ is a disc of radius r centered at z_0 and such that $\Delta(z_0,r) \subset U$, note that

$$(6) \qquad |f(z_0)| \leqq \frac{1}{\sqrt{\pi}\, r} \|f\|_U .$$

(For writing $z = z_0 + \rho e^{i\theta} \in \Delta(z_0,r)$, $0 \leq \rho < r$, the Cauchy integral formula reads

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z_0 + \rho e^{i\theta})d\theta ,$$

hence

$$\frac{r^2}{2} f(z_0) = \int_{\rho=0}^{r} f(z_0)\cdot\rho d\rho = \frac{1}{2\pi} \int_{\rho=0}^{r} \int_{\theta=0}^{2\pi} f(z_0 + \rho e^{i\theta})\rho d\rho d\theta ;$$

therefore

$$\begin{aligned} |f(z_0)| &\leqq \frac{1}{\pi r^2} \iint_{\Delta(z_0,r)} |f(z)| dx \wedge dy \\ &\leqq \frac{1}{\pi r^2} \left(\iint_{\Delta(z_0,r)} 1\, dx \wedge dy\right)^{\frac{1}{2}} \left(\iint_{\Delta(z_0,r)} |f(z)|^2 dx \wedge dy\right)^{\frac{1}{2}} \\ &\leqq \frac{1}{\sqrt{\pi}\, r} \|f\|_U , \end{aligned}$$

as desired.) Now it follows readily from this that $\Gamma_0(U, \mathcal{O})$ is complete in the above norm, hence that $\Gamma_0(U, \mathcal{O})$ is a Hilbert space.

To see this, consider any Cauchy sequence of elements $f_n \in \Gamma_0(U, \mathcal{O})$. For any compact subset $K \subset U$, select a number $r > 0$ so small that $\Delta(z,r) \subset U$ for any point $z \in K$; then by (6), $|f_m(z) - f_n(z)| \leq \frac{1}{\sqrt{\pi}\, r} \|f_m - f_n\|_U$ for all points $z \in K$, so that the sequence $\{f_n(z)\}$ is uniformly Cauchy on K, hence is uniformly convergent on K. The limit function $f(z) = \lim_n f_n(z)$ is then holomorphic in U, and is the limit of the sequence f_n in norm as well.

If $U \subset V \subset \mathbf{C}$ are two connected open subsets of the complex line, the restriction mapping

$$\rho_U : \Gamma_0(V, \mathcal{O}) \longrightarrow \Gamma_0(U, \mathcal{O})$$

is a bounded linear operator between these Hilbert spaces. Further, if $\bar{U}$ is compact and $\bar{U} \subset V$, this restriction mapping is even a compact operator, (that is, takes any bounded subset of $\Gamma_0(V, \mathcal{O})$ to a subset of $\Gamma_0(U, \mathcal{O})$ having compact closure). This is actually merely a restatement of Vitali's theorem; for if $\{f_n\}$ is any bounded sequence in $\Gamma_0(V, \mathcal{O})$, it follows from (6) that the functions $\{f_n(z)\}$ are uniformly bounded on the compact subset $\bar{U} \subset V$, and hence by Vitali's theorem there will be a subsequence which is uniformly convergent on $\bar{U}$.

To apply these considerations to the problem at hand, select a line bundle $\xi \in H^1(M, \mathcal{O}^*)$ over the Riemann surface M. Let $\mathfrak{U} = \{U_\alpha\}$ be a coordinate covering of M in terms of which the line bundle ξ is representable; that is, let $\mathfrak{U}$ be an open covering of M such that:

$$(7)\ \begin{cases} \text{(i) each set } U_\alpha \text{ is contained within a single coordinate neighborhood;} \\ \text{(ii) the line bundle } \xi \text{ has a representation cocycle } (\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^*) . \end{cases}$$

For any simplex $\sigma \in N(\mathfrak{U})$, then, $\Gamma(|\sigma|, \mathcal{O}(\xi)) \cong \Gamma(|\sigma|, \mathcal{O})$; and $|\sigma|$ is contained within a single coordinate neighborhood. Selecting any fixed coordinate mapping for the set $|\sigma|$, one can then introduce the subspace $\Gamma_0(|\sigma|, \mathcal{O}(\xi)) \subset \Gamma(|\sigma|, \mathcal{O}(\xi))$ of square-integrable sections of $\mathcal{O}(\xi)$ over $|\sigma|$; and the direct sum of these spaces, over all q-dimensional simplices $\sigma \in N(\mathfrak{U})$, is a subgroup $C_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \subset C^q(\mathfrak{U}, \mathcal{O}(\xi))$ which will be called the group of <u>square-integrable cochains</u> of $\mathfrak{U}$ with coefficients in $\mathcal{O}(\xi)$. Since the coboundary operator involves only restrictions and finite summations, it is clear that

$$\delta\colon C_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \longrightarrow C_0^{q+1}(\mathfrak{U}, \mathcal{O}(\xi)) ;$$

introducing the kernel $Z_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \subset C_0^q(\mathfrak{U}, \mathcal{O}(\xi))$ of the coboundary mapping, the corresponding square-integrable cohomology groups are defined by

$$H_0^q(\mathfrak{U}, \mathcal{O}(\xi)) = Z_0^q(\mathfrak{U}, \mathcal{O}(\xi))/\delta C_0^{q-1}(\mathfrak{U}, \mathcal{O}(\xi)) .$$

Notice that the inclusion $i\colon C_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \longrightarrow C^q(\mathfrak{U}, \mathcal{O}(\xi))$ induces a homomorphism

$$i^*\colon H_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \longrightarrow H^q(\mathfrak{U}, \mathcal{O}(\xi)) .$$

Lemma 8. Let $\mathfrak{U} = \{U_\alpha\}$ and $\mathfrak{V} = \{V_\alpha\}$ be open coverings of M, where $\mathfrak{U}$ is a refinement of $\mathfrak{V}$ such that $\overline{U}_\alpha \subset V_\alpha$; and suppose that both coverings satisfy condition (7). Then, for dimensions $q = 0$ and $q = 1$,

$$H_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \cong H^q(\mathfrak{U}, \mathcal{O}(\xi)) \cong H^q(M, \mathcal{O}(\xi)) .$$

Proof. First, observe that the homomorphism $i^*: H_0^q(\mathfrak{U}, \mathcal{O}(\xi)) \longrightarrow H^q(\mathfrak{U}, \mathcal{O}(\xi))$ is an injection (isomorphism into) for dimensions $q = 0, 1$. This is entirely obvious for $q = 0$. For the case $q = 1$, select a cocycle $(f_{\alpha\beta}) \in Z_0^1(\mathfrak{U}, \mathcal{O}(\xi))$, and suppose that $(f_{\alpha\beta})$ is cohomologous to zero in $H^1(\mathfrak{U}, \mathcal{O}(\xi))$, that is, that $f_{\alpha\beta} = f_\alpha - f_\beta$ for $(f_\alpha) \in C^1(\mathfrak{U}, \mathcal{O}(\xi))$; it suffices to show that each f_α is square-integrable over U_α. For any point $p \in \partial U_\alpha$, select an open set U_β such that $p \in U_\beta$; thus in $U_\alpha \cap U_\beta$, $f_\alpha = f_{\alpha\beta} + f_\beta$. However, the functions $f_{\alpha\beta}$ and f_β are both square-integrable in the intersection of U_α with some open neighborhood of p in M; hence f_α is square-integrable there as well. Since $\overline{U}_\alpha$ is compact, finitely many of these neighborhoods cover ∂U_α, and so clearly f_α is square-integrable in U_α, as desired.

Now, since each simplex σ of either covering $\mathfrak{U}$ or $\mathfrak{V}$ lies within a single coordinate neighborhood, and on $|\sigma|$ there is the isomorphism $\mathcal{O}(\xi) \cong \mathcal{O}$, it follows from the corollary to Theorem 4 that $H^q(|\sigma|, \mathcal{O}(\xi)) = 0$ for all dimensions $q \geq 1$; hence, by Theorem 5, $H^q(\mathfrak{U}, \mathcal{O}(\xi)) \cong H^q(\mathfrak{V}, \mathcal{O}(\xi)) \cong H^q(M, \mathcal{O}(\xi))$, for all dimensions $q \geq 0$. Indeed, as in Corollary 1 of Theorem 5, the homomorphism $\mu^*: H^q(\mathcal{V}, \mathcal{O}(\xi)) \to H^q(\mathcal{U}, \mathcal{O}(\xi))$ induced by the refinement $\mu: \mathcal{U} \to \mathcal{V}$ is an isomorphism. Note that for any simplex $\sigma \in N(\mathcal{U})$, its support $|\sigma|$ lies within a compact subset of $|\mu(\sigma)|$; so clearly the mapping μ^* can be factored as follows:

$$H^q(\mathcal{V}, \mathcal{O}(\xi)) \xrightarrow{\mu_0^*} H_0^q(\mathcal{U}, \mathcal{O}(\xi)) \xrightarrow{i^*} H^q(\mathcal{U}, \mathcal{O}(\xi)) .$$

Since $\mu^* = i^*\mu_0^*$ is an isomorphism, and i^* is an injection in dimensions $q = 0, 1$, it follows that μ_0^* is then an isomorphism, and that concludes the proof.

Theorem 7. If M is a compact Riemann surface and $\xi \in H^1(M, \mathcal{O}^*)$, then the cohomology groups $H^q(M, \mathcal{O}(\xi))$ are finite-dimensional complex vector spaces for dimensions $q = 0$ and $q = 1$.

Proof. Let $\mathcal{U} = \{U_\alpha\}$, $\mathcal{V} = \{V_\alpha\}$, and $\mathcal{W} = \{W_\alpha\}$, be finite open coverings of M, where $\mathcal{U}$ is a refinement of $\mathcal{V}$ such that $\overline{U}_\alpha \subset V_\alpha$ and $\mathcal{V}$ is a refinement of $\mathcal{W}$ such that $\overline{V}_\alpha \subset W_\alpha$; and suppose that all three coverings satisfy condition (7). For any simplex σ in the nerve of any of these coverings, the space $\Gamma_0(|\sigma|, \mathcal{O}(\xi))$ has a Hilbert space structure; thus the square-integrable cochain groups are Hilbert spaces, since they are finite sums of Hilbert spaces. The coboundary map being continuous, the square-integrable cocycles are also Hilbert spaces; but one cannot immediately say the same thing for the cohomology groups, since the space $\delta C_0^{p-1}(\mathcal{U}, \mathcal{O}(\xi))$ of coboundaries is not yet known to be a closed subspace.

First consider the case of dimension $q = 0$. Since $H_0^0(\mathcal{U}, \mathcal{O}(\xi)) = Z_0^0(\mathcal{U}, \mathcal{O}(\xi))$, this cohomology group is a Hilbert space. The refinement mapping $\mu: \mathfrak{U} \rightarrow \mathfrak{V}$ induces a bounded linear operator $\mu^*: H_0^0(\mathfrak{V}, \mathcal{O}(\xi)) \rightarrow H_0^0(\mathfrak{U}, \mathcal{O}(\xi))$. It follows directly from Lemma 8 that μ^* is an isomorphism of complex vector spaces, hence by the open mapping theorem is an isomorphism of Hilbert spaces. On the other hand, as a consequence of Vitali's theorem as noted earlier, the mapping $\mu^*: \Gamma(V_\alpha, \mathcal{O}(\xi)) \rightarrow \Gamma(U_\alpha, \mathcal{O}(\xi))$ is a compact operator; and since the cochain groups are finite sums of these spaces, the cohomology homomorphism is also a compact operator. These two observations taken together show that $H_0^0(\mathfrak{U}, \mathcal{O}(\xi))$ is a locally compact Hilbert space, hence is necessarily finite-dimensional.

Next, for dimension $q = 1$, consider the Hilbert space $C_0^0(\mathfrak{U}, \mathcal{O}(\xi)) + Z_0^1(\mathfrak{V}, \mathcal{O}(\xi))$. Applying the coboundary mapping to the first factor and the refinement mapping μ to the second factor yields a homomorphism

(8) $$(\delta,\mu): C_0^0(\mathfrak{U}, \mathcal{O}(\xi)) + Z_0^1(\mathfrak{V}, \mathcal{O}(\xi)) \rightarrow Z_0^1(\mathfrak{U}, \mathcal{O}(\xi)) .$$

It follows from Lemma 8 that this mapping is surjective; for since $\mu^*: H_0^1(\mathfrak{V}, \mathcal{O}(\xi)) \rightarrow H_0^1(\mathfrak{U}, \mathcal{O}(\xi))$ is an isomorphism, any cocycle of the covering $\mathfrak{U}$ must be cohomologous to a cocycle of the covering $\mathfrak{V}$. On the other hand, as noted above, the refinement mapping μ alone is a compact opertor; so applying the zero mapping to the first factor and μ to the second factor yields a compact operator

(9) $$(0,\mu): C_0^0(\mathfrak{U}, \mathcal{O}(\xi)) + Z_0^1(\mathfrak{V}, \mathcal{O}(\xi)) \rightarrow Z_0^1(\mathfrak{U}, \mathcal{O}(\xi)) .$$

The difference $(\delta,\mu) - (0,\mu)$ of the two mappings in (8) and (9) is just the coboundary map $\delta: C_0^0(\mathfrak{U}, \mathcal{O}(\xi)) \rightarrow Z_0^1(\mathfrak{U}, \mathcal{O}(\xi))$; the desired result, that the quotient $Z_0^1(\mathfrak{N}, \mathcal{Q}(\xi))/\delta C_0^0(\mathfrak{N}, \mathcal{Q}(\xi))$ is a finite dimensional vector space, is then an immediate consequence of the following lemma.

Lemma 9. Let X and Y be Hilbert spaces, and $\varphi: X \longrightarrow Y$ and $\psi: X \longrightarrow Y$ be bounded linear operators such that φ is surjective and ψ is compact. Then $(\varphi - \psi): X \longrightarrow Y$ has closed range of finite codimension in Y.

Proof. (For some of the general properties of linear operators required here, see N. Dunford and J. T. Schwartz, Linear Operators, vol. I, (Interscience, 1958).) Let $\varphi^*: Y \longrightarrow X$, $\psi^*: Y \longrightarrow X$ be the adjoint mappings to φ and ψ, respectively; then φ^* is a one-one mapping with closed range, and ψ^* is a compact mapping. Note first that the kernel K of the map $\varphi^* - \psi^*$ is a finite-dimensional subspace of Y. For suppose that $\{y_n\}$ is any bounded sequence of elements of K, so that $(\varphi^* - \psi^*)(y_n) = 0$. Since ψ^* is compact, then after passing to a subsequence if necessary, the sequence $\psi^*(y_n)$ will converge; and therefore $\varphi^*(y_n) = \psi^*(y_n)$ converges. Since φ^* is one-one and has closed range, it is a homeomorphism between Y and its range, hence $\{y_n\}$ converges. This shows that K is locally compact, hence finite-dimensional.

Now, factoring out by K, we may assume that $\varphi^* - \psi^*$ is a one-one mapping; and we then show that $\varphi^* - \psi^*$ has closed range. Consider a sequence $\{y_n\}$ of elements of Y such that $(\varphi^* - \psi^*)(y_n) \longrightarrow x$. If $\{y_n\}$ has a bounded subsequence, then as above we may suppose that $\psi^*(y_n)$ converges; but then $\varphi^*(y_n) = (\varphi^* - \psi^*)(y_n) + \psi^*(y_n)$ converges, so again y_n converges to an element y, and $(\varphi^* - \psi^*)(y) = x$. If $\|y_n\| \longrightarrow \infty$, consider the elements $y_n' = y_n\|y_n\|^{-1}$; these elements have norm 1, and $(\varphi^* - \psi^*)(y_n') = \|y_n\|^{-1} \cdot (\varphi^* - \psi^*)y_n \longrightarrow 0$. Again we may suppose that $\psi^*(y_n')$ converge, so that $\varphi^*(y_n')$ and hence y_n' converge, say to an element y'. Now $\|y'\| = 1$, and $(\varphi^* - \psi^*)(y') = 0 = (\varphi^* - \psi^*)(0)$; this contradicts the assumption that $\varphi^* - \psi^*$ is one-one, and this second case cannot occur.

To conclude the proof of the lemma, since $\varphi^* - \psi^*$ has closed range, the same is true of $\varphi - \psi$; so the quotient space $Y' = Y/(\varphi - \psi)(X)$ is a Hilbert space. Now φ induces a mapping $\varphi': X \longrightarrow Y'$ which is surjective, and ψ induces a mapping $\psi': X \longrightarrow Y'$ which is compact; and since clearly $\varphi' = \psi'$, the space Y' is locally compact, so finite dimensional, which concludes the proof.

§5. Differential forms and Serre duality.

(a) Some familiarity with differential forms, at least the definitions and most elementary properties, will be assumed here. (As references for this material, among other books are: L. Auslander and R. E. MacKenzie, Introduction to Differentiable Manifolds, (McGraw-Hill, 1963); and S. Helgason, Differential Geometry and Symmetric Spaces, Chapter I, (Academic Press, 1962).) We shall only consider the case of two-dimensional manifolds M, and shall adopt the following notation. The sheaf of germs of complex-valued C^∞ differential forms of degree r will be denoted by $\mathcal{E}^r$; thus the vector space of such differential forms over an open subset $U \subset M$ is just the space $\Gamma(U, \mathcal{E}^r)$. For the case $r = 0$, since a differential form of degree 0 is a function, $\mathcal{E}^0 = \mathcal{C}^\infty$. Within any coordinate neighborhood U, note that $\mathcal{E}^1 \cong \mathcal{E}^0 + \mathcal{E}^0$, and $\mathcal{E}^2 \cong \mathcal{E}^0$, (considered as sheaves of abelian groups); for if x,y are local coordinates in the neighborhood U, and $p \in U$ is any point of U, then any element $\varphi \in \mathcal{E}^1_p$ can be written uniquely in the form $\varphi = f\,dx + g\,dy$ for some germs $f,g \in \mathcal{E}^0_p = \mathcal{C}^\infty_p$, and any element $\psi \in \mathcal{E}^2_p$ can be written uniquely in the form $\psi = h\,dx \wedge dy$ for some germ $h \in \mathcal{E}^0_p = \mathcal{C}^\infty_p$. Of course, $\mathcal{E}^r = 0$ whenever $r > 2$. These are clearly all fine sheaves over M.

The operation of exterior differentiation is a sheaf homomorphism $d: \mathcal{E}^r \longrightarrow \mathcal{E}^{r+1}$; and there arises the exact sequence of sheaves (the <u>de Rham sequence</u>)

(1) $$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \longrightarrow 0 ,$$

recalling again that M is 2-dimensional. Since the sheaves $\mathcal{E}^r$ are all fine, it follows immediately from Theorem 3 that

$$H^q(M,\mathbb{C}) \cong \ker d_q^*/\operatorname{im} d_{q-1}^* \quad \text{for all } q > 0 ,$$

where $d_q^*\colon\ \Gamma(M, \mathcal{E}^q) \longrightarrow \Gamma(M, \mathcal{E}^{q+1})$ is the homomorphism of sections induced by exterior differentiation; this is of course de Rham's Theorem. For the particular case $q = 2$, it takes the form

(2) $$H^2(M,\mathbb{C}) \cong \frac{\Gamma(M, \mathcal{E}^2)}{d\Gamma(M, \mathcal{E}^1)} .$$

If M is a compact 2-dimensional manifold, then as is known, $H^2(M,\mathbb{C}) \cong \mathbb{C}$; (see the topological appendix for further remarks). Using (2), this isomorphism can be described explicitly as follows. For any differential form $\varphi \in \Gamma(M, \mathcal{E}^2)$, the integral $\iint_M \varphi$ is a well-defined complex number; the mapping $\varphi \longrightarrow \iint_M \varphi$ is clearly a linear mapping from $\Gamma(M,\mathcal{E}^2)$ onto the complex numbers. If $\varphi = d\psi$, it follows from Stokes' theorem that $\iint_M \varphi = \iint_M d\psi = 0$, so that this mapping yields a well-defined linear homomorphism $\Gamma(M, \mathcal{E}^2)/d\Gamma(M, \mathcal{E}$ $\longrightarrow \mathbb{C}$, which must indeed be an isomorphism since both sides are one-dimensional complex vector spaces.

Now suppose that M has a complex structure; and select some coordinate mapping $z_\alpha = x_\alpha + iy_\alpha$ in an open neighborhood of a point $p \in M$. Writing $dz_\alpha = dx_\alpha + idy_\alpha$ and $d\bar{z}_\alpha = dx_\alpha - idy_\alpha$ furnishes a new basis for the $\mathcal{E}^0_p$-module $\mathcal{E}^1_p$, so that

$$(3) \qquad \mathcal{E}^1_p \cong \mathcal{E}^0_p \, dz_\alpha + \mathcal{E}^0_p \, d\bar{z}_\alpha \, .$$

If $z_\beta = x_\beta + iy_\beta$ is another complex analytic coordinate mapping defined in anopen neighborhood of p, then since $z_\alpha = f_{\alpha\beta}(z_\beta)$ is a holomorphic function of z_β it follows from the Cauchy-Riemann equations that

$$dz_\alpha = f'_{\alpha\beta} \cdot dz_\beta \quad , \quad d\bar{z}_\alpha = \bar{f}'_{\alpha\beta} \cdot d\bar{z}_\beta \; ;$$

and so,

$$\mathcal{E}^0_p \, dz_\alpha = \mathcal{E}^0_p \, f'_{\alpha\beta} \, dz_\beta \cong \mathcal{E}^0_p \, dz_\beta \, .$$

It follows that the splitting (3) is intrinsically defined, that is, is independent of the choice of local coordinate mapping. Writing $\mathcal{E}^{1,0}_p = \mathcal{E}^0_p \, dz_\alpha$ and $\mathcal{E}^{0,1}_p = \mathcal{E}^0_p \, d\bar{z}_\alpha$, the splitting (3) becomes

$$(3') \qquad \mathcal{E}^1_p \cong \mathcal{E}^{1,0}_p + \mathcal{E}^{0,1}_p \, .$$

Since $\mathcal{E}^2_p \cong \mathcal{E}^0_p \, dx_\alpha \wedge dy_\alpha = \mathcal{E}^0_p \, dz_\alpha \wedge d\bar{z}_\alpha$, we shall for the sake of uniformity write $\mathcal{E}^2_p = \mathcal{E}^{1,1}_p$, and similarly $\mathcal{E}^0_p = \mathcal{E}^{0,0}_p$. Thus $\mathcal{E}^{r,s}$ denotes the sheaf of germs of complex-valued differential forms of total degree $r + s$ involving r basis elements dz_α and s basis elements $d\bar{z}_\alpha$; of course here $r = 0$ or 1, but the situation in the case of several complex variables is quite parallel.

In terms of the splitting (3'), the exterior derivative $d: \mathcal{E}^0 \longrightarrow \mathcal{E}^1 = \mathcal{E}^{1,0} + \mathcal{E}^{0,1}$ can be split as a direct sum $d = \partial + \bar{\partial}$ where $\partial: \mathcal{E}^{0,0} \longrightarrow \mathcal{E}^{1,0}$ and $\bar{\partial}: \mathcal{E}^{0,0} \longrightarrow \mathcal{E}^{0,1}$. For a coordinate mapping $z_\alpha = x_\alpha + iy_\alpha$, and a function $f(x,y) = f(z)$, secure

$$df = \frac{\partial f}{\partial x_\alpha} dx_\alpha + \frac{\partial f}{\partial y_\alpha} dy_\alpha = \frac{\partial f}{\partial z_\alpha} dz_\alpha + \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha ,$$

where $\partial/\partial z_\alpha$ and $\partial/\partial \bar{z}_\alpha$ are the linear differential operators introduced in §3(e), (page 40); this is a straight forward verification which will be left to the reader. It thus follows that

$$\partial f = \frac{\partial f}{\partial z_\alpha} dz_\alpha \quad , \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha .$$

In a parallel manner, for a differential form $\omega = f_\alpha dz_\alpha + g_\alpha d\bar{z}_\alpha$ of degree 1 it follows that

$$d\omega = \frac{\partial f_\alpha}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_\alpha + \frac{\partial g_\alpha}{\partial z_\alpha} dz_\alpha \wedge d\bar{z}_\alpha = \bar{\partial}(f_\alpha dz_\alpha) + \partial(g_\alpha d\bar{z}_\alpha) .$$

The de Rham sequence (1) then splits as follows:

$$\begin{array}{ccccccccc} & & & & & \overset{\partial}{\nearrow} \ \mathcal{E}^{1,0} \ \overset{\bar{\partial}}{\searrow} & & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{E}^{0,0} & & \mathcal{E}^{1,1} & \longrightarrow & 0 . \\ & & & & & \underset{\bar{\partial}}{\searrow} \ \mathcal{E}^{0,1} \ \underset{\partial}{\nearrow} & & & \end{array}$$

It is interesting to look at the separate pieces of this splitting of the de Rham sequence more closely. First of course there is an exact sheaf sequence (the Dolbeault sequence)

$$(4) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}^{0,0} \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \longrightarrow 0 .$$

The exactnees follows immediately from Lemma 6, and (4) is really merely an invariant form of the exact sequence of page 42. Since all the sheaves $\mathcal{E}^{r,s}$ are fine, it again follows from Theorem 3 that $H^1(M, \mathcal{O}) \cong \Gamma(M, \mathcal{E}^{0,1})/\bar\partial\Gamma(M, \mathcal{E}^{0,0})$ and that $H^q(M, \mathcal{O}) = 0$ for $q \geq 2$; this is Dolbeault's Theorem, and is an invariant restatement of equation (4) of page 42. Next, there is an exact sequence of sheaves of the form

$$(5) \qquad 0 \longrightarrow \mathcal{O}^{1,0} \longrightarrow \mathcal{E}^{1,0} \xrightarrow{\bar\partial} \mathcal{E}^{1,1} \longrightarrow 0 ,$$

where $\mathcal{O}^{1,0} \subset \mathcal{E}^{1,0}$ is defined as the kernel of the homomorphism $\bar\partial$ in (5). Introducing a local coordinate mapping z_α in the neighborhood of a point $p \in M$, a germ of a differential form $\varphi = f_\alpha(z_\alpha)dz_\alpha \in \mathcal{E}^{1,0}_p$ belongs to the subsheaf $\mathcal{O}^{1,0}$ if and only if

$$0 = \bar\partial\varphi = (\partial f_\alpha/\partial \bar z_\alpha)d\bar z_\alpha \wedge dz_\alpha ,$$

that is, if and only if the function $f_\alpha(z_\alpha)$ is a holomorphic function near p . The sheaf $\mathcal{O}^{1,0}$ is therefore called the sheaf of germs of holomorphic differential forms of type (1,0) , or also the sheaf of germs of Abelian differentials; a section of this sheaf is a holomorphic differential form or an abelian differential. Note that for an abelian differential φ , $d\varphi = \partial\varphi + \bar{\partial}\varphi = 0$; thus every abelian differential is a closed differential form. These forms can be introduced in several complex variables as well, but in the higher-dimensional cases they are not automatically closed forms, which makes for further complications.

(b) Now select a complex line bundle $\xi \in H^1(M, \mathcal{O}^*)$; and recall the construction given in §4(b) for the sheaf $\mathcal{C}^\infty(\xi)$ of germs of C^∞ cross-sections of the complex line bundle ξ . It is clear that the parallel construction yields <u>sheaves</u> $\mathcal{E}^{r,s}(\xi)$ <u>of germs of</u> C^∞ <u>differential forms which are cross-sections of the line bundle</u> ξ . (To carry out the construction, let $\mathfrak{U} = \{U_\alpha\}$ be a basis for the open sets of M such that each U_α is a coordinate neighborhood and that the line bundle ξ can be represented by a one-cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$. To each open set U_α associate the additive group $\mathcal{S}_\alpha = \Gamma(U_\alpha, \mathcal{E}^{r,s})$; and to each inclusion relation $U_\alpha \subset U_\beta$ associate the group homomorphism $\rho_{\alpha\beta}: \mathcal{S}_\beta \longrightarrow \mathcal{S}_\alpha$ which takes a differential form $\varphi_\beta \in \Gamma(U_\beta, \mathcal{E}^{r,s})$ to the differential form $\rho_{\alpha\beta}\, \varphi_\beta \in \Gamma(U_\alpha, \mathcal{E}^{r,s})$ defined by

$$(\rho_{\alpha\beta}\, \varphi_\beta)(p) = \xi_{\alpha\beta}(p) \cdot \varphi_\beta(p) \quad \text{for} \quad p \in U_\alpha \subset U_\beta .$$

This defines a complete presheaf, whose associated sheaf is the sheaf $\mathcal{E}^{r,s}(\xi)$.) These are clearly fine sheaves.

The ordinary exterior derivative cannot be applied to these sheaves to obtain an analogue of the de Rham sequence, since exterior differentiation does not commute with multiplication by $\xi_{\alpha\beta}$. However there does arise an analogue of the Dolbeault sequence. For if $\varphi \in \mathcal{E}^{r,s}(\xi)_p$ at a point $p \in M$, then φ is represented in each coordinate neighborhood U_α containing p by a germ φ_α of a differential form of type (r,s) at p; and if $p \in U_\alpha \cap U_\beta$, then $\varphi_\alpha = \xi_{\alpha\beta}\,\varphi_\beta$. Now since the functions $\xi_{\alpha\beta}$ are holomorphic, that is satisfy $\bar\partial \xi_{\alpha\beta} = 0$, it follows that $\bar\partial\varphi_\alpha = \xi_{\alpha\beta}\,\bar\partial\varphi_\beta$; therefore $\{\bar\partial\varphi_\alpha\} = \bar\partial\varphi$ is a well-defined element of $\mathcal{E}^{r,s+1}(\xi)_p$. This leads to a sequence of sheaves (<u>the Dolbeault-Serre sequence</u>)

$$(6) \qquad 0 \longrightarrow \mathcal{O}(\xi) \longrightarrow \mathcal{E}^{0,0}(\xi) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(\xi) \longrightarrow 0 .$$

In a single coordinate neighborhood this sequence reduces to the Dolbeault sequence, hence (6) is an exact sheaf sequence. The following generalization of formula (4) of §3(e) then follows trivially.

<u>Theorem 8.</u> If M is any Riemann surface and $\xi \in H^1(M, \mathcal{O}^*)$ is a line bundle over M, then

$$H^1(M, \mathcal{O}(\xi)) \cong \Gamma(M, \mathcal{E}^{0,1}(\xi)) / \bar\partial\, \Gamma(M, \mathcal{E}^{0,0}(\xi)) ,$$
$$H^q(M, \mathcal{O}(\xi)) = 0 \quad \text{for} \quad q \geqq 2 .$$

Proof. Since the sheaves $\mathcal{E}^{r,s}(\xi)$ are fine, the Dolbeault-Serre sequence (6) is a fine resolution of the sheaf $\mathcal{O}(\xi)$; the desired result then follows immediately from Theorem 3.

There is one result of prime importance, which is the basis of the further study of compact Riemann surfaces; indeed, the core of the analytic side of the theory consists of this result (and Theorem 4 or its analogues).

Theorem 9. (Serre's Duality Theorem). Let M be a compact Riemann surface, and $\xi \in H^1(M, \mathcal{O}^*)$ be any complex line bundle over M. Then the vector spaces $H^1(M, \mathcal{O}(\xi))$ and $H^0(M, \mathcal{O}^{1,0}(\xi^{-1}))$ are canonically dual to one another, hence have the same dimension.

The proof of this theorem will be given in the next section, following Serre (Un Théorème de Dualité, Comm. Math. Helv. 29 (1955), 9-26). In fact, the techniques in the proof of the theorem are of quite a different sort than the applications, and the reader who is willing to take this theorem on faith can omit the proof entirely and pass on to the applications. It is perhaps of interest to indicate briefly here just what the duality actually is, though. If $\varphi \in \Gamma(M, \mathcal{E}^{0,1}(\xi))$ and $\psi \in \Gamma(M, \mathcal{E}^{1,0}(\xi^{-1}))$ are any two C^∞ cross-sections of their respective line bundles, then note that their exterior product $\varphi \wedge \psi \in \Gamma(M, \mathcal{E}^{1,1})$. For in any coordinate neighborhood U_α these sections are represented by a differential form φ_α of type $(0,1)$ and a differential form ψ_α of type $(1,0)$; and at a point $p \in U_\alpha \cap U_\beta$ these differential forms satisfy $\varphi_\alpha(p) = \xi_{\alpha\beta}(p)\, \varphi_\beta(p)$ and $\psi_\alpha(p) = \xi^{-1}_{\alpha\beta}(p)\, \psi_\beta(p)$. The products $\varphi_\alpha \wedge \psi_\alpha$ are then differential forms of type $(1,1)$, and $\varphi_\alpha(p) \wedge \psi_\alpha(p) = \varphi_\beta(p) \wedge \psi_\beta(p)$ for $p \in U_\alpha \cap U_\beta$; therefore the products $\varphi_\alpha \wedge \psi_\alpha$ define a global differential form of type $(1,1)$ on the manifold M. Since M is compact, the integral

$$(7) \qquad (\varphi, \psi) = \iint_M \varphi \wedge \psi$$

is a well-defined complex number; then (7) defines a bilinear mapping

$$\Gamma(M, \mathcal{E}^{0,1}(\xi)) \times \Gamma(M, \mathcal{E}^{1,0}(\xi^{-1})) \longrightarrow \mathbb{C} .$$

Now if $\varphi \in \bar{\partial}\,\Gamma(M, \mathcal{E}^{0,0}(\xi)) \subset \Gamma(M, \mathcal{E}^{0,1}(\xi))$, so that $\varphi = \bar{\partial} f$ where $f \in \Gamma(M, \mathcal{E}^{0,0}(\xi))$, and if $\psi \in \Gamma(M, \mathcal{O}^{1,0}(\xi^{-1})) \subset \Gamma(M, \mathcal{E}^{1,0}(\xi^{-1}))$, so that $\bar{\partial}\psi = 0$, then

$$(\varphi, \psi) = \iint_M \bar{\partial} f \wedge \psi = \iint_M \bar{\partial}(f\psi) = \iint_M d(f\psi) = 0 ;$$

for since M is compact, and since $f\psi \in \Gamma(M, \mathcal{E}^{1,0}) \subset \Gamma(M, \mathcal{E}^{1})$, it follows from Stokes' theorem that $\iint_M d(f\psi) = 0$. Therefore the pairing (7) leads to a pairing

$$\Gamma(M, \mathcal{E}^{0,1}(\xi)) / \bar{\partial}\,\Gamma(M, \mathcal{E}^{0,0}(\xi)) \times \Gamma(M, \mathcal{O}^{1,0}(\xi^{-1})) \longrightarrow \mathbb{C} .$$

By Theorem 8, $H^1(M, \mathcal{O}(\xi)) \cong \Gamma(M, \mathcal{E}^{0,1}(\xi)) / \bar{\partial}\,\Gamma(M, \mathcal{E}^{0,0}(\xi))$; and as always, $H^0(M, \mathcal{O}^{1,0}(\xi^{-1})) \cong \Gamma(M, \mathcal{O}^{1,0}(\xi^{-1}))$. This therefore describes a bilinear pairing

$$H^1(M, \mathcal{O}(\xi)) \times H^0(M, \mathcal{O}^{1,0}(\xi^{-1})) \longrightarrow \mathbb{C} .$$

The assertion of Serre's theorem is that this is a dual or nonsingular pairing, hence that the two spaces are dual vector spaces. The spaces are then isomorphic as complex vector spaces; but whereas the above duality is canonical, the isomorphism is not. For most of the applications, what is required is merely that the two vector spaces have the same dimension.

It should be remarked in passing that the Serre duality theorem holds for higher-dimensional manifolds as well, in the sense that $H^q(M, \mathcal{O}(\xi))$ and $H^{n-q}(M, \mathcal{O}^{n,0}(\xi^{-1}))$ are dual on any 2n-dimensional compact complex manifold M for any integer $0 \leq q \leq n$. The greater strength of the theorem in the case of Riemann surfaces lies in the fact that all questions can be expressed in terms of the zero-dimensional cohomology groups in that case; in higher dimensions, one is faced with the problems of handling the cohomology groups $H^q(M, \mathcal{O}(\xi))$ for $q = 1,2,\ldots,[\frac{n}{2}]$ as well.

(c) The Serre duality theorem can be expressed without explicit mention of differential forms, by observing that the differential forms involved can be considered as cross-sections of line bundles themselves. This introduces a particularly useful line bundle, defined intrinsically on any Riemann surface as follows. Let $\mathcal{U} = \{U_\alpha\}$ be a complex analytic coordinate covering of the Riemann surface, with coordinate mappings $z_\alpha: U_\alpha \longrightarrow \mathbb{C}$. The coordinate transition functions $f_{\alpha\beta}$ are complex analytic local homeomorphisms between open subsets of $\mathbb{C}$, such that $z_\alpha(p) = f_{\alpha\beta}(z_\beta(p))$ for all $p \in U_\alpha \cap U_\beta$. Now introduce the functions $\kappa_{\alpha\beta}$ defined in the intersections $U_\alpha \cap U_\beta$ by

$$\kappa_{\alpha\beta}(p) = [f'_{\alpha\beta}(z_\beta(p))]^{-1} ;$$

since the functions $f'_{\alpha\beta}$ are holomorphic and nowhere-vanishing, the same is true of the functions $\kappa_{\alpha\beta}$. Furthermore, if $p \in U_\alpha \cap U_\beta \cap U_\gamma$ then $z_\alpha(p) = f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma(p))) = f_{\alpha\gamma}(z_\gamma(p))$, so that by the chain rule

$$\kappa_{\alpha\gamma}(p) = [f'_{\alpha\gamma}(z_\gamma(p))]^{-1} = [f'_{\alpha\beta}(f_{\beta\gamma}(z_\gamma(p))) \cdot f'_{\beta\gamma}(z_\gamma(p))]^{-1}$$

$$= \kappa_{\alpha\beta}(p) \cdot \kappa_{\beta\gamma}(p) ;$$

thus $(\kappa_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$. The element $\kappa \in H^1(M, \mathcal{O}^*)$ defined by this cocycle is called the <u>canonical line bundle</u> on the surface. Note that this bundle is independent of the choice of covering, since it can be constructed for a maximal covering of the surface M.

Now consider the sheaf $\mathcal{O}^{1,0}$ of germs of Abelian differentials on M. In terms of a coordinate covering $\mathfrak{U} = \{U_\alpha\}$, with coordinate mappings z_α: $U_\alpha \longrightarrow \mathbb{C}$, an element $\varphi \in \mathcal{O}^{1,0}_p$ is represented by a germ of a differential form $\varphi_\alpha = f_\alpha \cdot dz_\alpha$ in each coordinate neighborhood U_α containing p. If $p \in U_\alpha \cap U_\beta$, then $f_\alpha \cdot dz_\alpha = f_\beta \cdot dz_\beta$, so that $f_\alpha = dz_\beta/dz_\alpha \cdot f_\beta = \kappa_{\alpha\beta} \cdot f_\beta$; thus the coefficients f_α can be considered as elements of the sheaf $\mathcal{O}(\kappa)$. This then establishes an isomorphism $\mathcal{O}^{1,0} \cong \mathcal{O}(\kappa)$. In a completely parallel manner there is an isomorphism $\mathcal{O}^{1,0}(\xi) \cong \mathcal{O}(\kappa\xi)$ for any line bundle ξ, where the product $\kappa\xi$ is taken in the group $H^1(M, \mathcal{O}^*)$. Considering C^∞ rather than holomorphic sheaves, there is the isomorphism $\mathcal{E}^{1,0}(\xi) \cong \mathcal{E}^{0,0}(\kappa\xi) = \mathcal{C}^{\infty}(\kappa\xi)$. (The sheaf $\mathcal{E}^{0,1}(\xi)$ can be considered as isomorphic to the sheaf $\mathcal{C}^{\infty}(\bar{\kappa}\xi)$, when $\bar{\kappa}$ denotes the complex conjugate of the canonical bundle, in the obvious sense.)

In these terms, the Serre duality theorem can be restated as follows.

Theorem 9'. Let M be a compact Riemann surface, and $\xi \in H^1(M, \mathcal{O}^*)$ be any complex line bundle over M . Then the vector spaces $H^1(M, \mathcal{O}(\xi))$ and $H^0(M, \mathcal{O}(\kappa\xi^{-1}))$ are canonically dual to one another.

§6. Proof of Serre's duality theorem.*

(a) The proof will require some rather simple results about distributions; for the benefit of those not too familiar with distributions, we begin with a brief but self-contained review of those results which will be needed. The first part of this discussion will be restricted to subdomains U of the complex line $\mathbb{C}$, with the coordinates $z = x + iy$. For a function $f \in \mathcal{E}_U^\infty$, the support of f is the point set closure in U of the set $\{z \in U | f(z) \neq 0\}$; the support is thus a relatively closed subset of U, which will be denoted supp f. The subset of $\mathcal{E}_U^\infty$ consisting of those functions having compact support will be denoted by ${}_o\mathcal{E}_U^\infty$. To simplify notation, derivatives of functions $f \in \mathcal{E}_U^\infty$ will be denoted by

$$D^\nu f = D^{(\nu_1,\nu_2)}(f) = \frac{\partial^{\nu_1+\nu_2} f}{\partial x^{\nu_1} \partial y^{\nu_2}}, \quad \text{where } \nu = (\nu_1,\nu_2) .$$

Definition. A distribution in U is a linear mapping $T: {}_o\mathcal{E}_U^\infty \longrightarrow \mathbb{C}$, such that for every compact subset $K \subset U$ there are constants M and n with the property that

$$(1) \qquad |T(f)| \leq M \sum_{\nu_1+\nu_2 \leq n} \sup_{z \in K} |D^\nu f(z)| \quad \text{when } \operatorname{supp}(f) \subset K .$$

If the integer n can be chosen independently of K, the least possible value is called the order of the distribution. The set of all distributions in U is a linear space which will be denoted by $\mathcal{K}_U$.

*This section can be omitted on first reading, or omitted altogether by readers willing to take the Serre duality theorem on faith.

As an example, suppose that g is a Lebesgue measurable function in U which is integrable over any compact subset of U; there is an associated distribution T_g defined by

$$T_g(f) = \int_U f(z)g(z)dx \wedge dy \quad \text{for } f \in {}_o\mathcal{E}_U^\infty .$$

It is clear that this is actually a distribution, indeed, a distribution of order zero. In particular, the space $\mathcal{E}_U^\infty$ of infinitely differentiable functions is thus naturally imbedded as a subspace $\mathcal{E}_U^\infty \subset \mathcal{K}_U$. As another example, to any point $a \in U$ there is associated a distribution δ_a, the <u>Dirac distribution</u> centered at a, defined by

$$\delta_a(f) = f(a) \quad \text{for } f \in {}_o\mathcal{E}_U^\infty .$$

This is also clearly a distribution of order zero; and one sees thus that the space of distributions is properly larger than the space of locally integrable functions.

If $T \in \mathcal{K}_U$ and $g \in \mathcal{E}_U^\infty$, the <u>product</u> $gT \in \mathcal{K}_U$ is the distribution defined by

$$(gT)(f) = T(fg) \quad \text{for } f \in {}_o\mathcal{E}_U^\infty .$$

It is obvious that this is a distribution, and that it has order at most that of T if T is a distribution of finite order. Note also that whenever $g,h \in \mathcal{E}_U^\infty$, then $T_{gh} = g\cdot T_h = h\cdot T_g$; thus the product $\mathcal{E}_U^\infty \times \mathcal{K}_U \longrightarrow \mathcal{K}_U$ is compatible with the ordinary product of functions, on the subset $\mathcal{E}_U^\infty \subset \mathcal{K}_U$. However, this cannot be extended to an associative product on the full space $\mathcal{K}_U$ of distributions.

If $T \in \mathcal{K}_U$, the derivatives of that distribution are defined by

$$\begin{cases} \frac{\partial T}{\partial x}(f) = -T\left(\frac{\partial f}{\partial x}\right) \\ \frac{\partial T}{\partial y}(f) = -T\left(\frac{\partial f}{\partial y}\right) \end{cases} \quad \text{for } f \in {}_o\mathcal{E}_U^\infty .$$

It is also obvious that these derivatives are distributions, and that $\partial T/\partial x$ and $\partial T/\partial y$ are distributions of order at most $n+1$ if T is a distribution of order n; higher derivatives are defined inductively. Also, the linear partial differential operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ introduced in §3(e) can be applied to distributions as well as to functions. It should be observed that this definition is compatible with the usual notion of differentiation on the subspace $\mathcal{E}_U^\infty \subset \mathcal{K}_U$. For if $f \in {}_o\mathcal{E}_U^\infty$ and $g \in \mathcal{E}_U^\infty$, then

$$\frac{\partial T_g}{\partial x}(f) = -T_g \frac{\partial f}{\partial x} = -\int_U g(z) \frac{\partial f(z)}{\partial x} dx \wedge dy$$

$$= -\int_U \frac{\partial}{\partial x}(g(z)f(z))dx \wedge dy + \int_U f(z) \frac{\partial g(z)}{\partial x} dx \wedge dy .$$

Since $\operatorname{supp}(gf)$ is a compact subset of U, it follows from Stokes' theorem that $\int_U \partial(gf)/\partial x \, dx \wedge dy = 0$; and therefore

$$\frac{\partial T_g}{\partial x}(f) = \int_U f(z) \frac{\partial g(z)}{\partial x} dx \wedge dy = T_{\partial g/\partial x}(f) ,$$

as desired. The same result holds also for $\partial/\partial y$ and for all higher derivatives as well. This observation can be used to give a meaning to derivatives of arbitrary locally integrable functions, considered as distributions; and in fact, all distributions arise in this manner. Note that Leibniz' rule holds for differentiation of the product of a C^∞ function and a distribution.

If $V \subset U$ are two subdomains of the complex line, then clearly ${}_0\mathcal{E}^{\infty}_V \subset {}_0\mathcal{E}^{\infty}_U$; for every function $f \in {}_0\mathcal{E}^{\infty}_V$ can be extended to a function $f \in {}_0\mathcal{E}^{\infty}_U$ by setting it identically zero in $U - V$. Then any linear functional T defined on ${}_0\mathcal{E}^{\infty}_U$ defines a linear functional $\rho_{VU}T$ on ${}_0\mathcal{E}^{\infty}_V$ by restriction. In particular, if $T \in \mathcal{K}_U$ it is obvious that the restriction $\rho_{VU}T$ is a distribution in V. The restriction mapping is thus a homomorphism $\rho_{VU}: \mathcal{K}_U \longrightarrow \mathcal{K}_V$; and it is clear that $\rho_{WV}\rho_{VU} = \rho_{WU}$ wherever $W \subset V \subset U$. Thus if $\mathcal{U} = \{U\}$ is a basis for the open sets in the topology of $\mathbb{C}$, then the set $\{\mathcal{U}, \mathcal{K}_U, \rho_{UV}\}$ defines a presheaf over $\mathbb{C}$; the associated sheaf will be denoted by $\mathcal{K}$, and will be called the <u>sheaf of germs of distributions</u> over $\mathbb{C}$.

<u>Lemma 10.</u> The presheaf of distributions is a complete presheaf; hence there is a natural identification $\Gamma(U, \mathcal{K}) \cong \mathcal{K}_U$ for any open subset U.

Proof. Let U be a fixed open subset of $\mathbb{C}$, and let $\{U_\alpha\}$ be an open covering of U. Recalling the definition of a complete presheaf (page 19), there are two assertions to be proved.

First, suppose that $S, T \in \mathcal{K}_U$ are distributions such that $\rho_{U_\alpha U}S = \rho_{U_\alpha U}T$ for all U_α; then it must be shown that $S = T$. By passing to a refinement of the covering if necessary, there is no loss of generality in assuming that $\{U_\alpha\}$ is locally finite. Let $\{r_\alpha\}$ be a C^{∞} partition of unity subordinate to the covering $\{U_\alpha\}$, with $\operatorname{supp} r_\alpha$ compact for each α. For any $f \in {}_0\mathcal{E}^{\infty}_U$, write $f = \Sigma_\alpha r_\alpha f$; since $\operatorname{supp} f$ is compact, only finitely many terms of this series do not vanish identically. Thus since the sum is finite, and $\text{supp}(r_\alpha f) \subset U_\alpha$, it follows that $S(f) = S(\Sigma_\alpha r_\alpha f) = \Sigma_\alpha S(r_\alpha f) = \Sigma_\alpha T(r_\alpha f) = T(\Sigma_\alpha r_\alpha f) = T(f)$, which suffices for the desired result.

Second, suppose that $T_\alpha \in \mathcal{K}_{U_\alpha}$ are distributions such that $\rho_{U_\alpha \cap U_\beta, U_\alpha} T_\alpha = \rho_{U_\alpha \cap U_\beta, U_\beta} T_\beta$ for all $U_\alpha \cap U_\beta \neq \emptyset$; then it must be shown that there is a distribution $T \in \mathcal{K}_U$ such that $\rho_{U_\alpha U} T = T_\alpha$ for each α. Again assume that $\{U_\alpha\}$ is a locally finite covering, and select a subordinate partition of unity $\{r_\alpha\}$ with compact supports; so that any $f \in {}_o\mathcal{E}_U^\infty$ can be written $f = \Sigma_\alpha r_\alpha f$, where the sum is a finite sum. Define a linear functional $T: {}_o\mathcal{E}_U^\infty \longrightarrow \mathbb{C}$ by setting

$$T(f) = \Sigma_\alpha T_\alpha(r_\alpha f) \qquad \text{for} \quad f \in {}_o\mathcal{E}_U^\infty .$$

On the one hand, note that $\rho_{U_\beta U} T = T_\beta$; for if $f \in {}_o\mathcal{E}_U^\infty$ is such that $\text{supp}\, f \subset U_\beta$, then since $\text{supp}(r_\alpha f) \subset U_\alpha \cap U_\beta$ it follows that $T(f) = \Sigma_\alpha T_\alpha(r_\alpha f) = \Sigma_\alpha(\rho_{U_\alpha \cap U_\beta, U_\alpha} T_\alpha)(r_\alpha f) = \Sigma_\alpha(\rho_{U_\alpha \cap U_\beta, U_\beta} T_\beta)(r_\alpha f) = \Sigma_\alpha T_\beta(r_\alpha f) = T_\beta(f)$. On the other hand, observe that T is actually a distribution. For given a compact set $K \subset U$ and a function $f \in {}_o\mathcal{E}_U^\infty$ with $\text{supp}\, f \subset K$, since the sets $K_\alpha = K \cap \text{supp}\, r_\alpha \subset U_\alpha$ are compact and the T_α are distributions, it follows that

$$\begin{aligned} |T(f)| &\leq \Sigma_\alpha |T_\alpha(r_\alpha f)| \\ &\leq \Sigma_\alpha M_\alpha \sum_{\nu_1+\nu_2 \leq n_\alpha} \sup_{z \in K_\alpha} |D^\nu(r_\alpha f)(z)| ; \end{aligned}$$

the set of indices α in the above summation is actually finite, and depends only on the set K, so the above inequality clearly reduces to an inequality of the form (1), when $n = \max n_\alpha$ and M is suitably chosen. Then T is a distribution, and the proof is concluded.

The support of a distribution $T \in \mathcal{K}_U$ is defined to be the set of points in U which have no open neighborhood to which the restriction of T is the zero distribution; the support will be denoted by $\operatorname{supp} T$, and is clearly a relatively closed subset of U. Note that for a function $g \in \mathcal{C}^\infty_U$, $\operatorname{supp} g = \operatorname{supp} T_g$; and note further that when $T \in \mathcal{K}_U$ and $g \in \mathcal{C}^\infty_U$, $\operatorname{supp}(gT) \subset \operatorname{supp} g \cap \operatorname{supp} T$ Then multiplication of distributions by a C^∞ partition of unity defines a partition of unity in the sheaf $\mathcal{K}$; that is, <u>the sheaf of germs of distributions is a fine sheaf.</u>

(b) In a sense, the Cauchy-Riemann conditions hold for distributions as well as for differentiable functions. Before turning to the proof of this assertion, a few further simple properties of distributions are required.

<u>Lemma 11.</u> Suppose that $g(z,t)$ is a C^∞ function in $\mathbb{C} \times \mathbb{R}$, and that for any number t in an open interval $I \subset \mathbb{R}$ the support of $g(z,t)$ as a function of z alone is contained in a fixed compact set K. Then if T is a distribution in an open neighborhood U of K, the function $Tg(z,t)$ of t is a C^∞ function in the interval I.

Proof. For any point $t \in I$ and any $h \neq 0$, note that

(2) $$\frac{1}{h}[Tg(z,t+h) - Tg(z,t)] = T\left[\frac{g(z,t+h) - g(z,t)}{h}\right].$$

As $h \longrightarrow 0$, for a fixed value t, the function $[g(z,t+h) - g(z,t)]/h$ as well as its partial derivatives of any order with respect to x and y, converge uniformly on K; and their supports are always contained in K. It then follows immediately from the definition of a distribution that the expression (2) approaches $T[\partial g(z,t)/\partial t]$. Therefore $Tg(z,t)$ is a differentiable function of t, and its derivative is

$$\frac{d}{dt} Tg(z,t) = T\left[\frac{\partial g(z,t)}{\partial t}\right] .$$

Repeating the argument, the function $Tg(z,t)$ is thus C^∞ as a function of t in the interval I.

<u>Lemma 12.</u> Suppose that $G(z,\zeta)$ is a C^∞ function in $\mathbb{C} \times \mathbb{C}$, and that $\operatorname{supp} G \subset K \times L$ where $K, L \subset \mathbb{C}$ are compact sets. Then if T is a distribution in an open neighborhood U of K,

$$(3) \qquad T \iint_{\mathbb{C}} G(z,\zeta)d\zeta \wedge d\bar{\zeta} = \iint_{\mathbb{C}} TG(z,\zeta)d\zeta \wedge d\bar{\zeta} .$$

Proof. Note that $\iint_{\mathbb{C}} G(z,\zeta)d\zeta \wedge d\bar{\zeta}$ is a C^∞ function of z with support contained in K, and that $TG(z,\zeta)$ is a C^∞ function of ζ by Lemma 11, and has its support within L; therefore both sides in (3) are well defined. The Riemann sums for the integral $\iint_{\mathbb{C}} G(z,\zeta)d\zeta \wedge d\bar{\zeta}$ are all C^∞ functions of z with support contained in K; and these sums, as well as their partial derivatives of any order with respect to x and y, converge uniformly on K. Then (3) follows again immediately from the definition of a distribution.

With these properties out of the way, the Cauchy-Riemann conditions for a distribution read as follows; recall that the holomorphic functions can be considered as imbedded in the space of distributions, by associating to a holomorphic function h the distribution T_h.

<u>Theorem 10.</u> If T is a distribution in a subset $U \subset \mathbb{C}$ such that $\partial T/\partial \bar{z} = 0$, then T is a holomorphic function in U.

Proof. For any constant $\epsilon > 0$ let $U_\epsilon \subset U$ be the subset of U consisting of points whose distance from $\mathbb{C} - U$ is at least ϵ; and select a C^∞ function $r = r_\epsilon$ in $\mathbb{C}$ such that

$$r(z) = 1 \text{ for } |z| < \epsilon/2, \qquad r(z) = 0 \text{ for } |z| > \epsilon .$$

Then to any function $f \in {}_o\mathcal{C}^\infty_U$ with $\operatorname{supp} f \subset U_\epsilon$ associate the function

$$h(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} f(z+\zeta)\, \frac{r(\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta} ;$$

this function is clearly C^∞ everywhere, $\operatorname{supp} h \subset U$, and

$$2\pi i \frac{\partial h}{\partial \bar{z}} = \iint_{\mathbb{C}} \frac{\partial f(z+\zeta)}{\partial \bar{z}}\, \frac{r(\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta} = \iint_{\mathbb{C}} \frac{\partial f(z+\zeta)}{\partial \bar{\zeta}}\, \frac{r(\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta} .$$

For a fixed point $z \in \mathbb{C}$ let Δ_δ be a disc of radius δ centered at z; then

$$\begin{aligned} 2\pi i \frac{\partial h}{\partial \bar{z}} &= \lim_{\delta \to 0} \iint_{\mathbb{C} - \Delta_\delta} \frac{\partial f(z+\zeta)}{\partial \bar{\zeta}}\, \frac{r(\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta} \\ &= \lim_{\delta \to 0} \iint_{\mathbb{C} - \Delta_\delta} \frac{\partial}{\partial \bar{\zeta}} \left(f(z+\zeta)\, \frac{r(\zeta)}{\zeta}\, d\zeta \wedge d\bar{\zeta}\right) \\ &\quad - \lim_{\delta \to 0} \iint_{\mathbb{C} - \Delta_\delta} f(z+\zeta)\, \frac{\partial}{\partial \bar{\zeta}} \left(\frac{r(\zeta)}{\zeta}\right) d\zeta \wedge d\bar{\zeta} \\ &= 2\pi i\, f(z) - \iint_{\mathbb{C}} f(z+\zeta)\, \frac{\partial}{\partial \bar{\zeta}} \left(\frac{r(\zeta)}{\zeta}\right) d\zeta \wedge d\bar{\zeta} , \end{aligned}$$

as in the proof of Lemma 6. Note that the function

$$g_\epsilon(\zeta) = \begin{cases} \frac{1}{2\pi i} \frac{\partial}{\partial \bar\zeta}\left(\frac{r(\zeta)}{\zeta}\right) & \text{for } \zeta \neq 0 , \\ 0 & \text{for } \zeta = 0 , \end{cases}$$

is actually C^∞ everywhere, since $r(\zeta)/\zeta$ is holomorphic for $0 < |\zeta| < \epsilon/2$. Then write

$$f(z) = \frac{\partial h(z)}{\partial \bar z} + \iint_{\mathbb{C}} f(z+\zeta) g_\epsilon(\zeta)\, d\zeta \wedge d\bar\zeta$$

$$= \frac{\partial h(z)}{\partial \bar z} + \iint_{\mathbb{C}} g_\epsilon(\zeta - z) f(\zeta)\, d\zeta \wedge d\bar\zeta .$$

Since $\partial T/\partial \bar z = 0$, it follows that $T(\partial h/\partial \bar z) = -\frac{\partial T}{\partial \bar z}(h) = 0$; and so, applying Lemma 12,

$$Tf = \iint_{\mathbb{C}} Tg_\epsilon(\zeta - z) f(\zeta) d\zeta \wedge d\bar\zeta .$$

That is to say, the restriction of T to the set U_ϵ is the function $Tg_\epsilon(\zeta - z)$, which is a C^∞ function of ζ by Lemma 11. Since $\partial T/\partial \bar z = 0$ and T is a C^∞ function in U_ϵ , it follows from the ordinary Cauchy-Riemann conditions that T is a holomorphic function in U_ϵ . This holds for any value $\epsilon > 0$, hence T is indeed holomorphic throughout U , thus concluding the proof.

It is possible to continue in this vein, securing a fine resolution of the sheaf $\mathcal{O}$ by sheaves of germs of distributions, paralleling the discussion of §3(e); in particular, for any subset $M \subset \mathbb{C}$,

$$H^1(M, \mathcal{O}) \cong \Gamma(M, \mathcal{K}) \Big/ \frac{\partial}{\partial \bar z} \Gamma(M, \mathcal{K}) .$$

We shall not need this, so pursue the matter no further here.

(c) In order to extend the discussion of distributions to Riemann surfaces, it is first necessary to discuss the transformation properties of distributions. Suppose that U,V are subdomains of the complex line $\mathbb{C}$, and that $h: U \longrightarrow V$ is a C^∞ homeomorphism. The mapping h induces a linear mapping $h^*: \mathcal{G}^\infty_V \longrightarrow \mathcal{G}^\infty_U$, defined by $h^*(g) = g \circ h$, the composition of the two functions g and h. It is clearly of interest to extend the mapping h^* to a linear mapping $h^*: \mathcal{K}_V \longrightarrow \mathcal{K}_U$, recalling again the natural imbedding $\mathcal{G}^\infty_U \subset \mathcal{K}_U$ which associates to a function $g \in \mathcal{G}^\infty_U$ the distribution $T_g \in \mathcal{K}_U$. For this purpose, define a linear mapping $h^*: \mathcal{K}_V \longrightarrow \mathcal{K}_U$ by

(4) $$(h^*T)(f) = T[(f \circ h^{-1})J_h^{-1}] ,$$

where $T \in \mathcal{K}_V$, $f \in \mathcal{G}^\infty_U$, and J_h is the Jacobian determinant of the mapping h. Write $z = x+iy$ for a point in U and $\zeta = \xi + i\eta$ for a point in V, so that the mapping is of the form $\zeta = h(z)$. Then for any functions $g \in \mathcal{G}^\infty_V$ and $f \in {}_o\mathcal{G}^\infty_U$ note that

$$\begin{aligned}(h^*T_g)(f) &= T_g[(f \circ h^{-1})J_h^{-1}] \\ &= \iint_{\zeta \in V} g(\zeta)f(h^{-1}(\zeta)) \frac{\partial(x,y)}{\partial(\zeta,n)} d\xi \wedge d\eta \\ &= \iint_{z \in U} g(h(z))f(z)\, dx \wedge dy \\ &= T_{h^*(g)}(f) .\end{aligned}$$

Therefore the mapping h^* on $\mathcal{K}_V$, when restricted to the subspace $\mathcal{G}^\infty_V \subset \mathcal{K}_V$, coincides with the earlier definition of h^* on that subspace. It is a straightforward verification that when $g \in \mathcal{G}^\infty_V$ and $T \in \mathcal{K}_V$ then

(5) $$h^*(gT) = h^*(g)\cdot h^*(T)\ ;$$

and that when $k\colon V \longrightarrow W$ is another C^∞ homeomorphism and $T \in \mathcal{K}_W$ then

(6) $$(k\circ h)^*T = h^*(k^*T)\ .$$

The details will be left to the reader.

Now let M be a Riemann surface, with a complex analytic coordinate covering $\{U_\alpha, z_\alpha\}$ and coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$. A distribution T on the coordinate covering $\{U_\alpha, z_\alpha\}$ is defined to be a collection $\{T_\alpha\}$ of distributions on the various subsets $z_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}$ such that for each non-empty intersection $U_\alpha \cap U_\beta \subset M$,

(7) $$f^*_{\alpha\beta}(\rho_{z_\alpha(U_\alpha \cap U_\beta),V_\alpha}T_\alpha) = \rho_{z_\beta(U_\alpha \cap U_\beta),V_\beta}T_\beta\ .$$

Two distributions T and T' on coordinate coverings $\{U_\alpha, z_\alpha\}$ and $\{U'_\alpha, z'_\alpha\}$ are called equivalent if they define a distribution on the union of those coordinate coverings; that this is an equivalence relation in the proper sense is a consequence of (6). An equivalence class of distributions on coordinate coverings of M is defined to be a distribution on the Riemann surface. The sheaf $\mathcal{K}$ of germs of distributions is then a well-defined sheaf of abelian groups on M , and by Lemma 10 the global sections of the sheaf $\mathcal{K}$ are precisely the distributions on the Riemann surface. The natural imbedding of the C^∞ functions in the distributions exhibits $\mathcal{E}^\infty \subset \mathcal{K}$ as a subsheaf, in view of the remarks above. Furthermore, for any line bundle $\xi \in H^0(M, \mathcal{O}^*)$, the corresponding sheaf $\mathcal{K}(\xi)$ of germs of distribution cross-sections of the line bundle ξ can be constructed, paralleling the discussion of §4(b); for that construction merely requires that the multiplication of local sections of $\mathcal{K}$ by C^∞ functions be well defined, and so using (5) there are no difficulties. Details will be left to the reader. Adopting the notation of §5(c), we shall write $\mathcal{K}^{1,0}(\xi) = \mathcal{K}(\kappa\xi)$, $\mathcal{K}^{0,1}(\xi) = \mathcal{K}(\bar{\kappa}\xi)$, and $\mathcal{K}^{1,1}(\xi) = \mathcal{K}(\kappa\bar{\kappa}\xi)$, where $\kappa \in H^1(M, \mathcal{O}^*)$ is the canonical bundle on the surface.

The description of the sheaf $\mathcal{K}$ can be put in a somewhat more invariant form as follows. Again let $\{U_\alpha, z_\alpha\}$ be a coordinate covering of M, with $V_\alpha = z_\alpha(U_\alpha)$ and with coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$; note that the Jacobian determinant of the mapping $f_{\alpha\beta}$ is

$$J_{f_{\alpha\beta}} = \frac{\partial(x_\alpha, y_\alpha)}{\partial(x_\beta, y_\beta)} = \left| \frac{dz_\alpha}{dz_\beta} \right|^2 = |\kappa_{\beta\alpha}|^2 ,$$

where κ is the canonical bundle of M. Suppose that $\varphi \in \Gamma(M, \mathcal{E}^{1,1}) = \Gamma(M, \mathcal{E}^\infty(|\kappa|^2))$, and that $T \in \Gamma(M, \mathcal{K})$. Thus φ corresponds to a family of functions $\varphi_\alpha \in \Gamma(V_\alpha, \mathcal{E}^\infty)$ such that $f^*_{\beta\alpha}(\varphi_\beta) = |\kappa_{\beta\alpha}|^2 \varphi_\alpha$ in each intersection $U_\alpha \cap U_\beta \neq \emptyset$; and T corresponds to a family of distributions $T_\alpha \in \Gamma(V_\alpha, \mathcal{K})$ such that $f^*_{\alpha\beta}(T_\alpha) = T_\beta$ in each intersection $U_\alpha \cap U_\beta \neq \emptyset$. (The precise meaning of the phrase "in each intersection $U_\alpha \cap U_\beta$" is of course given by equation (7); the additional notation is cumbersome and of no great necessity, so will be dropped.) If $\operatorname{supp} \varphi \subset U_\alpha \cap U_\beta$, then

$$T_\beta(\varphi_\beta) = (f^*_{\alpha\beta} T_\alpha)(\varphi_\beta) = T_\alpha\left[(\varphi_\beta \circ f_{\beta\alpha}) J^{-1}_{f_{\alpha\beta}}\right] = T_\alpha\left(f^*_{\beta\alpha}(\varphi_\beta) J^{-1}_{f_{\alpha\beta}}\right)$$

$$= T_\alpha\left(|\kappa_{\beta\alpha}|^2 \varphi_\alpha \cdot |\kappa_{\beta\alpha}|^{-2}\right) = T_\alpha(\varphi_\alpha) ,$$

so that $T(\varphi)$ is well defined. More generally, let $\{r_\alpha\}$ be a C^∞ partition of unity on the manifold M, and suppose that $\operatorname{supp}\varphi$ is a compact subset of M; then define

$$(8) \qquad T(\varphi) = \Sigma_\alpha T_\alpha(r_\alpha\varphi) .$$

There are at most finitely many values of α for which $T_\alpha(r_\alpha\varphi_\alpha) \neq 0$, so that only a finite sum is involved; and as above, the sum is readily seen to be independent of the choice of partition of unity. (To see this, suppose that $\{s_\alpha\}$ is another such partition of unity. Then $\Sigma_\alpha T_\alpha(r_\alpha\varphi_\alpha) = \Sigma_{\alpha\beta} T_\alpha(r_\alpha s_\beta\varphi_\alpha) = \Sigma_{\alpha\beta} T_\beta(r_\alpha s_\beta\varphi_\beta) = \Sigma_\beta T_\beta(s_\beta\varphi_\beta)$, since $\operatorname{supp}(r_\alpha s_\beta\varphi_\alpha) \subset U_\alpha \cap U_\beta$.) Therefore the elements of $\Gamma(M, \mathcal{K})$ can be envisaged as linear functionals on the subspace ${}_0\Gamma(M, \mathcal{E}^{1,1}) \subset \Gamma(M, \mathcal{E}^{1,1})$ consisting of differential forms of compact support. In particular, if $g \in \Gamma(M, \mathcal{E}^\infty)$ and T_g is the associated distribution, then $T_g(\varphi) = \iint_M g\varphi$. It is thus clear why the distributions must be considered as linear functionals on the space of differential forms of degree 2 rather than on the space of functions; for this integral has an invariant meaning only in this case. In a precisely parallel manner, for any line bundle $\xi \in H^1(M, \mathcal{O}^*)$ the space $\Gamma(M, \mathcal{K}(\xi^{-1}))$ can be viewed as a space of linear functionals on ${}_0\Gamma(M, \mathcal{E}^{1,1}(\xi))$; the details will be left to the reader. The latter assertion can of course be rewritten, so that the space $\Gamma(M, \mathcal{K}^{1,1}(\xi^{-1}))$ can be viewed as a space of linear functionals on ${}_0\Gamma(M, \mathcal{E}^\infty(\xi))$; or yet again, the space $\Gamma(M, \mathcal{K}^{1,0}(\xi^{-1}))$ can be viewed as a space of linear functionals on ${}_0\Gamma(M, \mathcal{E}^{0,1}(\xi))$. If the distributions rather than the functions are required to have compact support, the corresponding assertions clearly hold; so that

${}_{o}\Gamma(M, \mathcal{K}(\xi^{-1}))$ can be viewed as a space of linear functions on $\Gamma(M, \mathcal{E}^{1,1}(\xi))$, and so on. If the Riemann surface M is compact, then ${}_{o}\Gamma(M, \mathcal{B}^{\infty}(\xi)) = \Gamma(M, \mathcal{C}^{\infty}(\xi))$ and ${}_{o}\Gamma(M, \mathcal{K}(\xi)) = \Gamma(M, \mathcal{K}(\xi))$.

One can, in fact, say considerably more. To simplify this discussion, suppose that M is a compact Riemann surface, and that $\{U_\alpha, z_\alpha\}$ is a finite coordinate covering of M with the property that the mapping z_α can be extended to be a C^∞ homeomorphism of an open neighborhood of $\bar{U}_\alpha$ into $\mathbf{C}$; and put $V_\alpha = z_\alpha(U_\alpha) \subset \mathbf{C}$ as before. For any integer $n \geq 0$ and any element $f = \{f_\alpha\} \in \Gamma(M, \mathcal{C}^{\infty}(\xi))$ put

(9) $$p_n(f) = \sum_{\alpha} \sum_{\nu_1+\nu_2 \leq n} \sup_{z_\alpha \in V_\alpha} |D^\nu f_\alpha(z_\alpha)| ,$$

with the notation as in §6(a). It is clear that the functions p_n so defined on $\Gamma(M, \mathcal{C}^{\infty}(\xi))$ are <u>norms</u>, in the sense that:

(i) $p_n(f) \geqq 0$, with equality holding only when $f = 0$;

(ii) $p_n(f+g) \leqq p_n(f) + p_n(g)$; and (iii) $p_n(cf) = |c| \cdot p^n(f)$ for any constant $c \in \mathbf{C}$. Introduce on $\Gamma(M, \mathcal{C}^{\infty}(\xi))$ the topology defined by this family of norms; so $\Gamma(M, \mathcal{C}^{\infty}(\xi))$ is a topological vector space, and a basis for the open neighborhoods of the zero element consists of sets of the form $\{f \in \Gamma(M, \mathcal{C}^{\infty}(\xi)) \mid |p_n(f)| < \epsilon$ for some $n\}$. (This makes $\Gamma(M, \mathcal{C}^{\infty}(\xi))$ into a Fréchet space; cf. C. Goffman and G. Pedrick, First Course in Functional Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1965.) A linear functional $T: \Gamma(M, \mathcal{C}^{\infty}(\xi)) \longrightarrow \mathbf{C}$ is continuous in this topology if and only if there are an integer n and a constant c such that $|T(f)| \leqq cp_n(f)$ for all $f \in \Gamma(M, \mathcal{C}^{\infty}(\xi))$.

Lemma 13. On a compact Riemann surface M, the space $\Gamma(M, \mathcal{K}^{1,1}(\xi^{-1}))$ is the space of continuous linear functionals on the topological vector space $\Gamma(M, \mathcal{E}^{\infty}(\xi))$.

Proof. If $T = \{T_\alpha\} \in \Gamma(M, \mathcal{K}^{1,1}(\xi))$, then T determines a linear functional on $\Gamma(M, \mathcal{E}^{\infty}(\xi))$ by (8), where $\{r_\alpha\}$ is any suitable partition of unity. Since T_α are distributions, the continuity of T follows immediately from a comparison of the definition of a distribution, (equation (1)), and the definition of the norms p_n, (equation (9)). Conversely, suppose that T is a continuous linear functional on $\Gamma(M, \mathcal{E}^{\infty}(\xi))$. Any function $f \in {}_o\mathcal{E}^{\infty}_{V_\alpha}$ can be extended to a section $f \in \Gamma(M, \mathcal{E}^{\infty}(\xi))$ by putting $f_\alpha(z_\alpha) = f(z_\alpha)$; and putting $f_\beta(z_\beta) = \xi^{-1}_{\alpha\beta}(z_\beta)f(f_{\alpha\beta}(z_\beta))$ for $z_\beta \in z_\beta(U_\alpha \cap U_\beta) \subset V_\beta$, extended to a function in V_β by setting it zero on the rest of V_β. Setting $T_\alpha(f) = T(f)$ defines a linear functional $T_\alpha\colon {}_o\mathcal{E}^{\infty}_{V_\alpha} \longrightarrow \mathbb{C}$; and it again follows readily from a comparison of (1) and (9) that T_α is a distribution in V_α. Note that if $f \in {}_o\mathcal{E}^{\infty}_{V_\alpha}$ has $\operatorname{supp} f \subset z_\alpha(U_\alpha \cap U_\beta)$, then $(f^*_{\beta\alpha}T_\beta)(f) = T_\beta[(f\cdot f_{\alpha\beta})\cdot|\kappa_{\beta\alpha}|^2] = |\kappa_{\beta\alpha}|^2\cdot T(g)$, where $g \in {}_o\mathcal{E}^{\infty}_{V_\beta}$ is defined by $g(z_\beta) = f(f_{\alpha\beta}(z_\beta))$. Since $g_\alpha(z_\alpha) = \xi^{-1}_{\beta\alpha}(z_\alpha)g(f_{\beta\alpha}(z_\alpha)) = \xi^{-1}_{\beta\alpha}(z_\alpha)f(z_\alpha) = \xi^{-1}_{\alpha\beta}(z_\alpha)f_\alpha(z_\alpha)$, it follows that $T(g) = \xi^{-1}_{\alpha\beta}T(f)$; therefore $f^*_{\beta\alpha}T_\beta = \xi^{-1}_{\alpha\beta}|\kappa_{\beta\alpha}|^2T_\alpha$, so that $\{T_\alpha\} \in \Gamma(M, \mathcal{K}^{1,1}(\xi^{-1}))$. Now for any $f \in \Gamma(M, \mathcal{E}^{\infty}(\xi))$ and any partition of unity $\{r_\alpha\}$ it follows that $T(f) = T(\Sigma_\alpha r_\alpha f) = \Sigma_\alpha T(r_\alpha f) = \Sigma_\alpha T_\alpha(r_\alpha f)$; this therefore identifies T with the section $\{T_\alpha\} \in \Gamma(M, \mathcal{K}^{1,1}(\xi^{-1}))$, and serves to conclude the proof.

(d) With these preparations out of the way, let us now turn to the proof of the Serre duality theorem. Recall that the statement of the theorem is as follows.

Theorem 9. Let M be a compact Riemann surface, and $\xi \in H^1(M, \mathcal{O}^*)$ be a complex line bundle over M. Then the vector spaces $H^1(M, \mathcal{O}(\xi))$ and $H^0(M, \mathcal{O}^{1,0}(\xi^{-1}))$ are canonically dual to one another.

Proof. Introduce the complex vector spaces

$$A = \Gamma(M, \mathcal{E}^{0,0}(\xi)) , \qquad B = \Gamma(M, \mathcal{E}^{0,1}(\xi)) ,$$

and the homomorphism $\bar{\partial}: A \longrightarrow B$, as in §5(b). It follows from Theorem 8 that $H^1(M, \mathcal{O}(\xi)) \cong B/\bar{\partial}A$, and from Theorem 7 that this quotient space is a finite-dimensional complex vector space. Further introduce the norms p_n given by (9), in both of the spaces A and B; this makes these spaces into Frechet spaces, and it is obvious that the homomorphism $\bar{\partial}: A \longrightarrow B$ is a continuous mapping in terms of this topology. Note further that the image $\bar{\partial}A \subset B$ is a closed subspace of B. (To see this, let $K \subset A$ be the kernel of the mapping $\bar{\partial}$, so that K is a closed subspace of A; thus A/K is also a Frechet space, and the mapping $\bar{\partial}: A/K \longrightarrow B$ is a continuous linear mapping with trivial kernel. Let $L \subset B$ be a finite-dimensional subspace of B which is complementary to $\bar{\partial}(A) = \bar{\partial}(A/K)$. The product $(A/K) \times L$ is then a Frechet space, and the mapping $\bar{\partial} + i: (A/K) \times L \longrightarrow B$ given by

$$(\bar{\partial} + i)(a, \ell) = \bar{\partial}(a) + \ell$$

is a continuous mapping. This mapping also has trivial kernel, and its image is all of B; so by the open mapping theorem, it is an isomorphism of Frechet spaces. Since $(A/K) \times 0 \subset (A/K) \times L$ is a closed subspace, it then follows that $(\bar{\partial} + i)(A/K \times 0) = \bar{\partial}A$ is a closed subspace of B, as desired.) The fact that $\bar{\partial}A$ is closed means that $B/\bar{\partial}A$ inherits from B a topology as a Frechet space, which must of course coincide with the usual topology of a finite dimensional vector space. The conclusion of interest here is then that any linear functional on the quotient space $B/\bar{\partial}A$, which is necessarily continuous since $B/\bar{\partial}A$ is finite-dimensional, leads to a <u>continuous</u> linear functional on B which vanishes on the subspace $\bar{\partial}A$.

Now, purely formally, let A^*, B^* be the dual spaces to the topological vector spaces A and B; and let $\bar{\partial}^* : B^* \longrightarrow A^*$ be the dual homomorphism to $\bar{\partial}$. It then follows readily that the kernel $K^* \subset B^*$ of the homomorphism $\bar{\partial}$ is the dual space to $B/\bar{\partial}A$. For on the one hand, any linear functional $\tilde{T}$ on $B/\bar{\partial}A$ determines a continuous linear functional T on B which vanishes on $\bar{\partial}A$, as noted above; and thus $T \in B^*$, indeed $T \in K^*$. Conversely if $T \in K^*$ then T defines a continuous linear functional on B which vanishes on $\bar{\partial}A$, hence a linear functional on $B/\bar{\partial}A$.

Then, to conclude the proof, we need merely appropriately identify the spaces A^*, B^*, and the homomorphism $\bar{\partial}^*$. It follows immediately from Lemma 13 that

$$A^* \cong \Gamma(M, \mathcal{K}^{1,1}(\xi^{-1})) , \qquad B^* \cong \Gamma(M, \mathcal{K}^{1,0}(\xi^{-1})) ;$$

and recalling the definition of the derivatives of a distribution, it is clear that $\bar{\partial}^{*'} = -\bar{\partial}$, where $\bar{\partial}$ is the familiar operator as applied to distributions. Now $B/\bar{\partial}A \cong H^1(M, \mathcal{O}(\xi))$ is therefore dual to the kernel of the mapping

$$-\bar{\partial}: \Gamma(M, \mathcal{K}^{1,0}(\xi^{-1})) \longrightarrow \Gamma(M, \mathcal{K}^{1,1}(\xi^{-1})) \ ;$$

but by Theorem 10 this kernel is precisely $\Gamma(M, \mathcal{O}^{1,0}(\xi^{-1})) =$ $= H^0(M, \mathcal{O}^{1,0}(\xi^{-1}))$, and thus the proof is concluded.

An examination of the proof indicates that the duality is indeed that described in §5(b); details will be left to the reader.

§7. Riemann-Roch theorem.

(a) Before turning to the Riemann-Roch theorem itself, it is necessary to introduce a fundamental invariant associated to any complex line bundle, its Chern class or characteristic class. This is actually the first step in the classification of complex line bundles, in the sense of providing a detailed description of the group $H^1(M, \mathcal{O}^*)$ over a compact Riemann surface M. Recall the exact sequence of sheaves (cf. §2(d))

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0 ,$$

where the homomorphism e was defined by $e(f) = \exp 2\pi i f$. The associated exact cohomology sequence includes the segment

$$H^1(M,\mathbb{Z}) \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(M, \mathcal{O}) .$$

Since $H^2(M, \mathcal{O}) = 0$ by Theorem 8, this sequence can be rewritten

(1) $$0 \longrightarrow H^1(M, \mathcal{O})/H^1(M,\mathbb{Z}) \longrightarrow H^1(M, \mathcal{O}^*) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow 0 .$$

The coboundary homomorphism in (1) will be called the <u>characteristic homomorphism</u> $c: H^1(M, \mathcal{O}^*) \longrightarrow H^2(M,\mathbb{Z})$; for a line bundle $\xi \in H^1(M, \mathcal{O}^*)$, the image $c(\xi)$ will be called the <u>characteristic class</u> or <u>Chern class</u> of the line bundle ξ. The sequence (1) goes a good deal of the way towards describing the group of line bundles in more detail; there remains the problem of investigating the group $H^1(M, \mathcal{O})/H^1(M,\mathbb{Z})$, and this will be tackled in a later section to complete this point.

In a sense, the Chern class measures the topological properties of the line bundle ξ. To make this precise, consider the sheaf $\mathcal{C}^{\infty *}$ of germs of C^∞ complex-valued, nowhere-vanishing functions on M, the group structure being multiplicative. The sheaf $\mathcal{O}^*$ is a subsheaf of $\mathcal{E}^{\infty*}$, the natural inclusion defining a sheaf homomorphism $i: \mathcal{O}^* \rightarrow \mathcal{E}^{\infty*}$; and also there is an exact sheaf sequence of the form $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{E}^{\infty} \rightarrow \mathcal{E}^{\infty*} \rightarrow 0$, paralleling the above sheaf sequence in the analytic case. Altogether, these two exact sequences can be written as parts of the commutative diagram of sheaves and sheaf homomorphisms, as follows.

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathcal{O} & \xrightarrow{e} & \mathcal{O}^* & \rightarrow & 0 \\
 & & \downarrow \cong & & \downarrow i & & \downarrow i & & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathcal{E}^{\infty} & \xrightarrow{e} & \mathcal{E}^{\infty*} & \rightarrow & 0
\end{array}$$

The cohomology sequences can then be written together as a commutative diagram o. groups and homomorphisms of the form

$$\begin{array}{ccccccc}
H^1(M, \mathcal{O}) & \rightarrow & H^1(M, \mathcal{O}^*) & \xrightarrow{c} & H^2(M, \mathbb{Z}) & \rightarrow & 0 \\
\downarrow i^* & & \downarrow i^* & & \downarrow \cong & & \\
H^1(M, \mathcal{E}^{\infty}) & \rightarrow & H^1(M, \mathcal{E}^{\infty*}) & \xrightarrow{c} & H^2(M, \mathbb{Z}) & \rightarrow & H^2(M, \mathcal{E}^{\infty})
\end{array}$$

The homomorphism c in the second line is the parallel to the characteristic homomorphism in the first line; but in the second line c is an isomorphism, since $H^1(M, \mathcal{E}^{\infty}) = H^2(M, \mathcal{E}^{\infty}) = 0$ because these sheaves are fine. Now if $\xi \in H^1(M, \mathcal{O}^*)$ is a line bundle such that $c(\xi) = 0$, the image $i^*(\xi) \in H^1(M, \mathcal{E}^{\infty*})$ will satisfy $ci^*(\xi) = c(\xi) = 0$ by commutativity; but since c is an isomorphism at that level, $i^*(\xi) = 0$. The converse holds as well, as is obvious, so that $c(\xi) = 0$ if and only if $i^*(\xi) = 0$. Now $i^*(\xi)$ is the topological form of the line bundle ξ, so that we may say that the line bundle is topologically trivial precisely when it has zero Chern class. Selecting a representative cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ for the line bundle ξ, the condition that $i^*(\xi) = 0$ is just that there exist nowhere-vanishing C^∞ functions f_α defined in the various sets U_α and such that $f_\beta(p)/f_\alpha(p) = \xi_{\alpha\beta}(p)$ for $p \in U_\alpha \cap U_\beta$, and the condition that $\xi = 0$ is that there exist holomorphic nowhere-vanishing such functions f_α; this observation may help to clarify the above discussion.

Since we shall henceforth assume that M is a compact two-dimensional manifold, it is known that $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$; the Chern class $c(\xi) \in H^2(M, \mathbb{Z})$ of a line bundle ξ can thus be considered as an integer, under the above identification. This identification of the Chern class as an integer can be made more explicit as follows. The class $c(\xi)$ can be considered as an element of the group $H^2(M, \mathbb{C})$; for either apply the cohomology homomorphism $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{C})$ derived from the inclusion mapping $\mathbb{Z} \subset \mathbb{C}$ of sheaves, or recall from the universal coefficient theorem that $H^2(M, \mathbb{C}) \cong H^2(M, \mathbb{Z}) \otimes \mathbb{C}$. Under the isomorphism $H^2(M, \mathbb{C}) \cong \Gamma(M, \mathcal{E}^2)/d\Gamma(M, \mathcal{E}^1)$ furnished by de Rham's theorem, the cohomology class $c(\xi) \in H^2(M, \mathbb{C})$ will be represented by a differential form $\varphi(\xi) \in \Gamma(M, \mathcal{E}^2)$; and then $\iint_M \varphi(\xi) \in \mathbb{C}$ will be the constant associated to that cohomology class under the identification $H^2(M, \mathbb{C}) \cong \mathbb{C}$ introduced in Section 5. In fact, this will be an integer, and will be called the Chern class of ξ and also denoted by $c(\xi)$. A useful explicit form for the Chern class in this sense is given as follows.

<u>Lemma 14.</u> Let $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ represent a line bundle ξ, and suppose that $\{r_\alpha\}$ are nowhere-vanishing C^∞ functions defined in the open sets U_α and satisfying $r_\alpha(p) = r_\beta(p)|\xi_{\beta\alpha}(p)|^2$

for $p \in U_\alpha \cap U_\beta$. Then $\varphi = \frac{1}{2\pi i}\partial\bar{\partial}\log r_\alpha \in \Gamma(M, \mathcal{E}^2)$ is a well-defined differential form on M , and

$$c(\xi) = \iint_M \varphi = \frac{1}{2\pi i}\iint_M \partial\bar{\partial}\log r_\alpha .$$

Proof. This is a straightforward matter of tracing through the identifications in the preceding paragraph, recalling the explicit form of the coboundary homomorphism of an exact sheaf sequence as given in the proof of Theorem 1. First, to pass from the line bundle to its characteristic class, consider the exact sheaf sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0$. The cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ will be the image under e of a cochain $(\sigma_{\alpha\beta}) \in C^1(\mathcal{U}, \mathcal{O})$; indeed, merely take $\sigma_{\alpha\beta} = \frac{1}{2\pi i}\log \xi_{\alpha\beta}$, for any fixed choice of a branch of the logarithm in each set $U_\alpha \cap U_\beta \neq \emptyset$; by suitably refining the covering, all such intersections can be taken to be simply connected, and the logarithms are thereby well-defined. The characteristic class $c(\xi) \in H^2(M,\mathbb{Z})$ is represented by the 2-cocycle given by the coboundary of the 1-cochain $(\sigma_{\alpha\beta})$, namely, the 2-cocycle $c_{\alpha\beta\gamma} \in Z^2(\mathcal{U},\mathbb{Z})$ where $c_{\alpha\beta\gamma} = \sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta} =$ $= \sigma_{\alpha\beta} + \sigma_{\beta\gamma} + \sigma_{\gamma\alpha}$. Now this 2-cocycle can be envisaged as belonging to the group $Z^2(\mathcal{U},\mathbb{C})$, and the homomorphisms given in de Rham's theorem follow from the exact sheaf sequence $0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \longrightarrow 0$. Introducing the subsheaf $\mathcal{E}^1_c \subset \mathcal{E}^1$ of closed differential forms, consider the exact sequence $0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1_c \longrightarrow 0$. The 2-cocycle $(c_{\alpha\beta\gamma})$ considered as an element of $Z^2(\mathcal{U}, \mathcal{E}^0)$ will be the coboundary of some 1-cochain $(\sigma'_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{E}^0)$; the condition is that $c_{\alpha\beta\gamma} = \sigma'_{\alpha\beta} + \sigma'_{\beta\gamma} + \sigma'_{\gamma\alpha}$. Then $(d\sigma'_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{E}^1_c)$. In fact, referring to the first part of the proof, we shall merely take $\sigma'_{\alpha\beta} = \sigma_{\alpha\beta}$, so that $d\sigma'_{\alpha\beta} = \frac{1}{2\pi i} d \log \xi_{\alpha\beta}$. Now from the exact sequence $0 \longrightarrow \mathcal{E}^1_c \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \longrightarrow 0$, the element $(d\sigma'_{\alpha\beta})$ considered as an element of $Z^1(\mathfrak{U}, \mathcal{E}^1)$ will be the coboundary of some 0-cochain $(\tau_\alpha) \in C^0(\mathfrak{U}, \mathcal{E}^1)$; the condition is that $d\sigma'_{\alpha\beta} = \tau_\beta - \tau_\alpha$. Then $d\tau_\alpha = d\tau_\beta$, so $(d\tau_\alpha)$ define a global differential form, a 1-cocycle of $Z^0(\mathfrak{U}, \mathcal{E}^2)$. To be explicit, selecting any C^∞ differential forms τ_α of degree 1 in the sets U_α such that

(2) $$\tau_\alpha = \frac{1}{2\pi i} d \log \xi_{\alpha\beta} + \tau_\beta \quad \text{in } U_\alpha \cap U_\beta .$$

The differential form $\varphi \in \Gamma(M, \mathcal{E}^2)$ defined by $\varphi = d\tau_\alpha$ in U_α is well-defined, and the Chern class (as an integer) is given by

$$c(\xi) = \iint_M d\varphi .$$

Now, to finish the proof of the lemma, the functions r_α are nowhere-vanishing, hence have well-defined logarithms; and these logarithms satisfy

$$\log r_\alpha(p) = \log r_\beta(p) + \log \xi_{\beta\alpha}(p) + \log \overline{\xi_{\beta\alpha}(p)} \quad \text{for } p \in U_\alpha \cap U_\beta .$$

Since the functions $\xi_{\beta\alpha}(p)$ are holomorphic, $\partial \log \xi_{\beta\alpha} = d \log \xi_{\beta\alpha}$ and $\partial \log \overline{\xi}_{\beta\alpha} = 0$; thus the differential forms $\tau_\alpha = \frac{i}{2\pi} \partial \log r_\alpha$ clearly satisfy (2). The differential form φ will then be given by $\varphi = d\tau_\alpha = \frac{i}{2\pi} d\partial \log r_\alpha = \frac{i}{2\pi} \bar{\partial}\partial \log r_\alpha$; and

$$c(\xi) = \frac{1}{2\pi i} \iint_M \partial\bar{\partial} \log r_\alpha ,$$

as desired.

<u>Remark.</u> There always exist functions $\{r_\alpha(p)\}$ having the properties required in the preceding lemma, and indeed the functions

can always be taken to have strictly positive values. For introducing the sheaf $\mathcal{R}^*$ of germs of positive-valued C^∞ functions, the collection $\{|\xi_{\alpha\beta}|^2\}$ defines a 1-cocycle in $Z^1(\mathcal{U}, \mathcal{R}^*)$, and the desired functions $\{r_\alpha\}$ form a zero-cochain having this 1-cocycle as its coboundary; thus it suffices to show that $H^1(M, \mathcal{R}^*) = 0$. Now the subsheaf $\mathcal{R} \subset \mathcal{C}^\infty$ of real-valued functions is clearly fine, and the ordinary exponential mapping $\exp: \mathcal{R} \longrightarrow \mathcal{R}^*$ is a sheaf isomorphism; hence $H^1(M, \mathcal{R}^*) \cong H^1(M, \mathcal{R}) = 0$, as desired.

Theorem 11. For any line bundle $\xi \in H^1(M, \mathcal{O}^*)$ on a compact Riemann surface M, and any non-trivial cross-section $f \in \Gamma(M, \mathcal{M}^*(\xi))$,

$$c(\xi) = \sum_{p \in M} \nu_p(f) ,$$

where $\nu_p(f)$ is the order of the cross-section f at the point $p \in M$, as defined in §4(b).

Proof. Since M is compact there are only finitely many points $p \in M$ at which $\nu_p(f) \neq 0$; calling these points p_i, the divisor of f has the form

$$\mathcal{J}(f) = \Sigma_i \nu_i \cdot p_i ,$$

and the assertion of the theorem is that $c(\xi) = \Sigma_i \nu_i$. Let $\mathcal{U} = \{U_\alpha\}$ be a coordinate covering of M such that the bundle ξ is represented by a cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$; and suppose that the covering is so chosen that each point p_i has an open neighborhood V_i for which $V_i \subset U_{\alpha_i}$ for some index α_i but $V_i \cap U_\alpha = \emptyset$ for $\alpha \neq \alpha_i$. The functions f_α representing the cross-section $f \in \Gamma(M, \mathcal{M}^*(\xi))$ are meromorphic in U_α and satisfy $f_\alpha = \xi_{\alpha\beta} f_\beta$

in $U_\alpha \cap U_\beta$. The functions $|f_\alpha|^2$ are thus C^∞ and nowhere-vanishing in $U_\alpha - (\cup_i p_i) \cap U_\alpha$, and satisfy $|f_\alpha|^2 = |\xi_{\alpha\beta}|^2 \cdot |f_\beta|^2$; and these functions can be modified arbitrarily within the sets V_i without changing the functional equations. It is thus evident that there are C^∞ , positive-valued functions g_α defined in the various sets U_α , such that

$$g_\alpha = |\xi_{\alpha\beta}|^2 g_\beta \quad \text{in } U_\alpha \cap U_\beta ,$$
$$g_\alpha = |f_\alpha|^2 \quad \text{in } U_\alpha - U_\alpha \cap \cup_i V_i .$$

By Lemma 14, the Chern class of the bundle ξ is given by

$$c(\xi) = \frac{1}{2\pi i} \iint_M \partial\bar\partial \log g_\alpha^{-1} = \frac{1}{2\pi i} \iint_M \bar\partial\partial \log g_\alpha .$$

Since $g_\alpha = |f_\alpha|^2$ on $M - \cup_i V_i$, and
$\bar\partial\partial \log g_\alpha = \bar\partial\partial(\log f_\alpha + \log \bar f_\alpha) = 0$ since f_α is holomorphic there, it follows that

$$c(\xi) = \frac{1}{2\pi i} \Sigma_i \iint_{V_i} \bar\partial\partial \log g_\alpha .$$

By Stokes' theorem, since $\bar\partial\partial \log g_\alpha = d\,\partial \log g_\alpha$, secure that

$$c(\xi) = \frac{1}{2\pi i} \Sigma_i \int_{\partial V_i} \partial \log g_\alpha$$

where ∂V_i is the boundary of V_i . Now $g_\alpha = f_\alpha$ on ∂V_i , so that actually

$$c(\xi) = \frac{1}{2\pi i} \Sigma_i \int_{\partial V_i} \partial \log f_\alpha = \frac{1}{2\pi i} \Sigma_i \int_{\partial V_i} d \log f_\alpha$$
$$= \Sigma_i \nu_i$$

by the residue theorem; this completes the proof.

It is an immediate consequence of the preceding that all meromorphic cross-sections of the bundle ξ have the same total order on the Riemann surface M , where the total order is by definition

$\Sigma_{p \in M} \nu_p(f)$. This can be taken as the definition of the characteristic class of a line bundle, since as we shall shortly see every bundle does have a non-trivial meromorphic cross-section. One further useful trivial consequence of this theorem is the following.

<u>Corollary.</u> If $\xi \in H^1(M, \mathcal{O}^*)$ is a line bundle on the compact Riemann surface M such that $c(\xi) < 0$, then there are no non-trivial cross-sections of the sheaf $\mathcal{O}(\xi)$, or equivalently, $\Gamma(M, \mathcal{O}(\xi)) = 0$.

Proof. If $f \in \Gamma(M, \mathcal{O}(\xi))$ and f is not identically zero, then $\Sigma_{p \in M} \nu_p(f) \geqq 0$; thus necessarily $c(\xi) \geqq 0$, by the theorem.

(b) Again suppose that M is a compact Riemann surface, and consider a line bundle $\xi \in H^1(M, \mathcal{O}^*)$. Introduce the expression

(3) $$\chi(\xi) = \dim H^0(M, \mathcal{O}(\xi)) - \dim H^1(M, \mathcal{O}(\xi)) - c(\xi) ;$$

the cohomology groups are finite-dimensional complex vector spaces, and the dimension is meant in that sense. Note that, applying the Serre duality theorem, this expression can also be written in the form

(3'.) $$\chi(\xi) = \dim \Gamma(M, \mathcal{O}(\xi)) - \dim \Gamma(M, \mathcal{O}(\kappa\xi^{-1})) - c(\xi) ,$$

where κ is the canonical bundle of the surface M . The content of the Riemann-Roch theorem is that this expression $\chi(\xi)$ is independent of the choice of the line bundle ξ . A first step in the proof of this assertion is the following.

Lemma 15. Let $\mathcal{D}$ be a divisor on the compact Riemann surface, and let $\eta = \delta^*\mathcal{D} \in H^1(M, \mathcal{O}^*)$ be the line bundle corresponding to that divisor, as in §4(b). Then for any line bundle $\xi \in H^1(M, \mathcal{O}^*)$,

$$\chi(\xi\eta) = \chi(\xi) .$$

Proof. Clearly it is sufficient to prove this assertion in the case that the divisor is a single point, say $\mathcal{D} = 1 \cdot q$. Paralleling the discussion in §4(b), introduce the subsheaf $\mathcal{O}_{\mathcal{M}}(\mathcal{D}, \xi) \subset \mathcal{M}(\xi)$ defined by

$$\mathcal{O}_{\mathcal{M}}(\mathcal{D}, \xi)_p = \{f \in \mathcal{M}(\xi)_p \mid \text{either } f \equiv 0 \text{ or } \mathcal{D}(f) \geqq \mathcal{D} \text{ near } p\} ;$$

since $\mathcal{D} = 1 \cdot q > 0$, actually $\mathcal{O}_{\mathcal{M}}(\mathcal{D}, \xi) \subset \mathcal{O}(\xi)$. The quotient sheaf $\mathcal{S} = \mathcal{O}(\xi)/\mathcal{O}_{\mathcal{M}}(\mathcal{D}, \xi)$ clearly has the form

$$\mathcal{S}_p = \begin{cases} 0 & \text{if } p \neq q \\ \mathbb{C} & \text{if } p = q \end{cases} ;$$

(compare with the example discussed on pages 23 and 24). As in Lemma 7, it follows that

$$\mathcal{O}_{\mathcal{M}}(\mathcal{D}, \xi) \cong \mathcal{O}(\xi\eta) .$$

There thus follows the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(\xi\eta) \longrightarrow \mathcal{O}(\xi) \longrightarrow \mathcal{S} \longrightarrow 0 .$$

Consider then the associated exact cohomology sequence:

$$(4) \qquad \begin{aligned} 0 \longrightarrow H^0(M, \mathcal{O}(\xi\eta)) \longrightarrow H^0(M, \mathcal{O}(\xi)) \longrightarrow H^0(M, \mathcal{S}) \longrightarrow \\ H^1(M, \mathcal{O}(\xi\eta)) \longrightarrow H^1(M, \mathcal{O}(\xi)) \longrightarrow H^1(M, \mathcal{S}) \longrightarrow \cdots . \end{aligned}$$

Since $\mathcal{S}$ is a skyscraper sheaf, having stalk $\mathbb{C}$ at a single point, it follows readily that $H^0(M, \mathcal{S}) \cong \mathbb{C}$ and $H^1(M, \mathcal{S}) = 0$. Now in an exact sequence of complex vector spaces as in (4), the alternating sum of the dimensions of the vector spaces is zero; this can be rewritten as the equality

$$\text{(5)} \qquad \begin{aligned} &\dim H^0(M, \mathcal{O}(\xi\eta)) - \dim H^1(M, \mathcal{O}(\xi\eta)) + 1 \\ &\quad = \dim H^0(M, \mathcal{O}(\xi)) - \dim H^1(M, \mathcal{O}(\xi)) . \end{aligned}$$

Note that $c(\eta) = -1$. (This peculiar observation results from the notational conventions adopted. Recalling Lemma 7, and its preceding discussion, the line bundle of a divisor $\mathcal{D}$ was defined to be the element $\delta^*(\mathcal{D}) \in H^1(M, \mathcal{O}^*)$ in the exact sequence (2) of §4. If (d_α) are local functions defining the divisor, then the functions $1/d_\alpha$ are a meromorphic cross-section of the sheaf $\mathcal{M}^*(\mathcal{D})$, by this convention. The total order of the divisor $\mathcal{D}$ is thus the <u>negative</u> of the total order of any meromorphic cross-section of its associated line bundle. Hence, for the line bundle η associated to the divisor $1\cdot q$, it follows from Theorem 11 that $c(\eta) = -1$.) Then, replacing 1 in equation (5) by $-c(\eta) = c(\eta^{-1})$, adding $c(\xi)$ to both sides of the equation, and recalling that $c(\xi) + c(\eta^{-1}) = c(\xi\eta^{-1})$, it follows that $\chi(\xi\eta^{-1}) = \chi(\xi)$, which suffices to conclude the proof.

Now by using this lemma and the Serre duality theorem, it is an easy matter to prove the fundamental existence theorems on a Riemann surface. The discussion in §4(b) should be recalled here.

<u>Theorem 12.</u> On a compact Riemann surface M , $H^1(M, \mathcal{M}^*) = 0$; equivalently, every line bundle on M has a non-trivial meromorphic cross-section, hence every line bundle is the bundle of a divisor.

Proof. The equivalence of the three assertions of the theorem was noted in §4(b). Noting that a line bundle is the bundle of a divisor on M precisely when the bundle admits a non-trivial meromorphic cross-section, it clearly suffices to show that, given any complex line bundle ξ, there exists a line bundle η of a divisor such that $\xi\eta$ has a non-trivial meromorphic cross-section. In fact, we shall show more, namely, that given a line bundle ξ, there exists a line bundle η of a divisor such that $\xi\eta$ has a non-trivial holomorphic cross-section, that is, such that $\Gamma(M, \mathcal{O}(\xi\eta)) \neq 0$.

Suppose, contrarily, that $\Gamma(M, \mathcal{O}(\xi\eta)) = 0$ for every line bundle η of a divisor on the surface M. By Lemma 15 the expression

$$\chi(\xi\eta) = \dim \Gamma(M, \mathcal{O}(\xi\eta)) - \dim \Gamma(M, \mathcal{O}(\kappa\xi^{-1}\eta^{-1})) - c(\xi\eta)$$

is independent of η; and since $\dim \Gamma(M, \mathcal{O}(\xi\eta)) = 0$ for all η by assumption, it follows that

$$\dim \Gamma(M, \mathcal{O}(\kappa\xi^{-1}\eta^{-1})) + c(\xi\eta) = C$$

is independent of η. There are bundles η with arbitrarily given characteristic classes $c(\eta)$; and by taking $c(\eta)$ to be large enough, $c(\kappa\xi^{-1}\eta^{-1}) = c(\kappa\xi^{-1}) - c(\eta) < 0$. Thus, by the Corollary to Theorem 11, it follows that $\dim \Gamma(M, \mathcal{O}(\kappa\xi^{-1}\eta^{-1})) = 0$ whenever $c(\eta)$ is large; but then $c(\xi\eta) = c(\xi) + c(\eta)$ would also be independent of η, which is absurd. This contradiction then proves the theorem.

This existence theorem then shows that the study of divisors on the surface can indeed be reduced to the study of line bundles. In particular, referring to equation (3) of page 53, the divisor class group $\underset{\sim}{A}(M)$ of the surface M is isomorphic to the group

$H^1(M, \mathcal{O}^*)$ of line bundles on the surface.

<u>Corollary.</u> On a compact Riemann surface M, the characteristic $\chi(\xi) = \dim H^0(M, \mathcal{O}(\xi)) - \dim H^1(M, \mathcal{O}(\xi)) - c(\xi)$ is a constant, independent of the choice of the line bundle ξ.

Proof. Since every line bundle ξ on the surface M is the line bundle of a divisor as a consequence of the theorem, it follows from Lemma 15 that $\chi(\xi) = \chi(1)$ for any ξ, which serves to prove the desired result.

It is of course of some importance to determine the constant $\chi(\xi)$ for a given surface M. For this purpose set $\xi = 1$, the trivial bundle, and note (using Serre duality) that

$$\begin{aligned} \chi(1) &= \dim \Gamma(M, \mathcal{O}) - \dim \Gamma(M, \mathcal{O}^{1,0}) - c(1) \\ &= 1 - \dim \Gamma(M, \mathcal{O}^{1,0}) . \end{aligned}$$

The constant

$$g = \dim \Gamma(M, \mathcal{O}^{1,0}) , \tag{6}$$

the dimension of the space of abelian differentials on the surface M, is caled the <u>genus</u> of the surface M. This constant has a simple topological interpretation as follows. Considering the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \mathcal{O}^{1,0} \longrightarrow 0 ,$$

the associated exact cohomology sequence has the form

$$\begin{aligned} 0 \longrightarrow H^0(M,\mathbb{C}) \longrightarrow H^0(M, \mathcal{O}) \longrightarrow H^0(M, \mathcal{O}^{1,0}) \longrightarrow H^1(M,\mathbb{C}) \longrightarrow \\ H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^{1,0}) \longrightarrow H^2(M,\mathbb{C}) \longrightarrow 0 , \end{aligned}$$

since $H^2(M, \mathcal{O}) = 0$ by Theorem 8. Now $H^0(M,\mathbb{C}) = H^0(M, \mathcal{O}) = \mathbb{C}$, since all the global holomorphic functions on a compact Riemann surface are constant; and $H^2(M,\mathbb{C}) \cong \mathbb{C}$ as noted earlier. Therefore, recalling that the alternating sum of the dimensions of the terms in a finite exact sequence of vector spaces is zero, it follows that

$$\dim H^0(M, \mathcal{O}^{1,0}) - \dim H^1(M,\mathbb{C}) + \dim H^1(M, \mathcal{O})$$
$$(7) \qquad - \dim H^1(M, \mathcal{O}^{1,0}) + 1 = 0 .$$

By definition, $\dim H^0(M, \mathcal{O}^{1,0}) = g$, and by the Serre duality theorem,

$$\dim H^1(M, \mathcal{O}) = \dim H^0(M, \mathcal{O}^{1,0}) = g , \quad \text{and}$$
$$\dim H^1(M, \mathcal{O}^{1,0}) = \dim H^0(M, \mathcal{O}) = 1 ;$$

therefore (7) becomes

$$(8) \qquad 2g = \dim H^1(M,\mathbb{C}) .$$

That is to say, $H^1(M,\mathbb{C})$ is an even-dimensional complex vector space, and its dimension is twice the genus of the surface M. The constant $\chi(\xi) = 1 - g$. As yet another interpretation of the genus, consider the canonical bundle κ; then

$$1 - g = \chi(\kappa) = \dim H^0(M, \mathcal{O}(\kappa)) - \dim H^1(M, \mathcal{O}(\kappa)) - c(\kappa) .$$

By definition, $\dim H^0(M, \mathcal{O}(\kappa)) = \dim H^0(M, \mathcal{O}^{1,0}) = g$; and by the Serre duality theorem, $\dim H^1(M, \mathcal{O}(\kappa)) = \dim H^1(M, \mathcal{O}^{1,0}) =$ $= \dim H^0(M, \mathcal{O}) = 1$. Therefore

$$(9) \qquad c(\kappa) = 2(g - 1) ,$$

relating the genus to the characteristic class of the canonical bundle. Since cross-sections of the canonical bundle are just abelian differentials, it follows from Theorem 11 that the total order of an abelian differential on M is precisely $2(g-1)$.

In terms of the genus, the Corollary to Theorem 12 can be restated as follows.

Theorem 13. (Riemann-Roch Theorem) If M is a compact Riemann surface of genus g and $\xi \in H^1(M, \mathcal{O}^*)$ is a complex line bundle on M , then

$$\dim H^0(M, \mathcal{O}(\xi)) - \dim H^1(M, \mathcal{O}(\xi)) - c(\xi) = 1 - g ;$$

or equivalently,

$$\dim \Gamma(M, \mathcal{O}(\xi)) - \dim \Gamma(M, \mathcal{O}(\kappa\xi^{-1})) - c(\xi) = 1 - g ,$$

where κ is the canonical bundle.

In some cases, the Riemann-Roch Theorem furnishes explicitly the dimension of the space of holomorphic cross-sections of a complex line bundle; the following table may prove useful in keeping this in mind. As a notational convenience, for a line bundle $\xi \in H^1(M, \mathcal{O}^*)$ we shall write

$$\gamma(\xi) = \dim \Gamma(M, \mathcal{O}(\xi)) . \tag{10}$$

Then:

$$(11) \begin{cases} \text{(a)} \quad c(\xi) < 0 \implies \gamma(\xi) = 0 \\ \text{(b)} \quad c(\xi) = 0 \implies \gamma(\xi) = \begin{cases} 1 & \text{if } \xi = 1 \\ 0 & \text{if } \xi \neq 1 \end{cases} \\ \text{(c)} \quad c(\xi) = 2g-2 \implies \gamma(\xi) = \begin{cases} g & \text{if } \xi = \kappa \\ g-1 & \text{if } \xi \neq \kappa \end{cases} \\ \text{(d)} \quad c(\xi) > 2g-2 \implies \gamma(\xi) = c(\xi) - (g-1) . \end{cases}$$

To see that (11) holds, recall first that $c(\xi) < 0$ implies that $\Gamma(M, \mathcal{O}(\xi)) = 0$, by the Corollary to Theorem 11. Furthermore, by that same corollary, if $c(\xi) = 0$ and $\gamma(\xi) > 0$ there must exist at least one non-trivial (i.e., not identically vanishing) holomorphic cross-section of the line bundle ξ, and its divisor must have total order $c(\xi) = 0$; that is, that cross-section must be holomorphic and nowhere-vanishing on M. This means that $\xi = 1$ (cf. page 56), and hence $\gamma(\xi) = 1$. Now when $c(\xi) = 2g-2$, recalling that $c(\kappa) = 2g-2$ and applying part (b) of (11) to the bundle $\kappa\xi^{-1}$, it follows that

$$\gamma(\kappa\xi^{-1}) = \begin{cases} 1 & \text{if} \quad \kappa\xi^{-1} = 1 \\ 0 & \text{if} \quad \kappa\xi^{-1} \neq 1 \end{cases}$$

Then by the Riemann-Roch Theorem, $\gamma(\xi) = \gamma(\kappa\xi^{-1}) + c(\xi) - (g-1) =$ $= \gamma(\kappa\xi^{-1}) + g-1$, from which (c) follows immediately. Finally, if $c(\xi) > 2g-2$, then $c(\kappa\xi^{-1}) = c(\kappa) - c(\xi) < 0$, so it follows from part (a) that $\gamma(\kappa\xi^{-1}) = 0$; and part (d) follows immediately from the Riemann-Roch Theorem again.

For the line bundles between the trivial bundle and the canonical bundle, that is, for those bundles $\xi \in H^1(M, \mathcal{O}^*)$ such that $0 < c(\xi) < 2g-2$, the Riemann-Roch Theorem merely provides the equality

$$\gamma(\xi) = \gamma(\kappa\xi^{-1}) + c(\xi) - (g-1) ;$$

and thus the formula merely relates two unknown quantities. However, it is easy to obtain some useful inequalities for line bundles in this range. First, since $\gamma(\kappa\xi^{-1}) \geqq 0$, it follows that

(12) $$\gamma(\xi) \geqq c(\xi) - (g-1) \quad \text{for all} \quad \xi \in H^1(M, \mathcal{O}^*) .$$

To obtain inequalities in the other direction, select a line bundle ζ with $c(\zeta) = 1$ and $\gamma(\zeta) \geqq 1$; for instance, the line bundle associated to the divisor $-1 \cdot p$ can be taken as ζ , since it has at least one holomorphic cross-section h . Now multiplying a cross-section of any line bundle ξ by the section h yields a cross-section of the line bundle $\xi\zeta$; and thus $\gamma(\xi\zeta) \geqq \gamma(\xi)$. Repeating this process, $\gamma(\xi\zeta^r) \geqq \gamma(\xi)$ for any index $r \geqq 0$. If $c(\xi) < 2g-2$, take in particular $r = 2g-1-c(\xi)$; then

$$c(\xi\zeta^r) = c(\xi) + r \cdot c(\zeta) = 2g-1 ,$$

so by (11d) it follows that

$$\gamma(\xi\zeta^r) = g .$$

It then follows that $\gamma(\xi) \leqq g$ whenever $c(\xi) < 2g-2$. If $0 < c(\xi) < 2g-2$, then $c(\kappa\xi^{-1}) < 2g-2$ so that $\gamma(\kappa\xi^{-1}) \leqq g$; and by the Riemann-Roch Theorem, $\gamma(\xi) = \gamma(\kappa\xi^{-1}) + c(\xi) - (g-1) \leqq c(\xi) + 1$. Consequently,

(13) $$\gamma(\xi) \leqq \min(g, c(\xi)+1) \quad \text{for} \quad 0 < c(\xi) < 2g-2 .$$

The following table may help to keep these inequalities in mind:

(14)

$c(\xi)$:	0	1	2	...	g-2	g-1	g	g+1	...	2g-3	2g-2	2g-1
max $\gamma(\xi)$:	1	2	3	...	g-1	g	g	g	...	g	g	g
min $\gamma(\xi)$:	0	0	0	...	0	0	1	2	...	g-2	g-1	g

We shall next see that for a general line bundle ξ the value $\gamma(\xi)$ is the minimum given in the table.

(c). At this stage it is possible to extend a bit further the discussion begun in §4, concerning relations between divisors and line bundles on a compact Riemann surface M. For any point $p \in M$ select a complex line bundle $\zeta_p \in H^1(M, \mathcal{O}^*)$, such that there is a section $f \in \Gamma(M, \mathcal{O}(\zeta_p))$ for which $\mathcal{J}(f) = 1 \cdot p$; there always exists such a bundle, since the line bundle associated to the divisor $-1 \cdot p$ under the exact sequence (2) in §4 will do. In fact, the bundle ζ_p is determined uniquely by the point p ; for if ζ_p' were any other such bundle, with $f' \in \Gamma(M, \mathcal{O}(\zeta_p'))$ its corresponding section, then $f'/f \in \Gamma(M, \mathcal{O}(\zeta_p'/\zeta_p))$ is a holomorphic nowhere-vanishing section, hence necessarily $\zeta_p'/\zeta_p = 1$. Line bundles of this form will be called <u>point bundles</u> on the surface M. Note that $c(\zeta_p) = 1$ for any point bundle ζ_p, by Theorem 11; and that $\gamma(\zeta_p) \geq 1$, since by definition a point bundle has at least one non-trivial section.

<u>Lemma 16.</u> If M is a compact Riemann surface of genus $g > 0$ and ζ_p is a point bundle on M, then $\gamma(\zeta_p) = 1$.

Proof. Suppose contrariwise that $\gamma(\zeta_p) \geqq 2$; of course, then, $\gamma(\zeta_p) = 2$ by the inequalities (14). Select two linearly independent sections $f_1, f_2 \in \Gamma(M, \mathcal{O}(\zeta_p))$, and consider the mapping $M \longrightarrow \mathbf{P}^1$ defined by

$$q \in M \longrightarrow (f_1(q), f_2(q)) \in \mathbf{P}^1 , \tag{15}$$

in terms of homogeneous coordinates on the projective line. Since the values of $f_1(q)$, $f_2(q)$ in two different coordinate neighborhoods are multiplied by the same non-zero constant when passing from one neighborhood to the other, these values determine the same point in the projective line. If both functions f_1, f_2 vanish at a point, then dividing by their greatest common divisor will yield the same mapping into projective space, thus extending that mapping to the common zero of the functions f_1, f_2. It is thus evident that (15) is a well-defined complex analytic mapping from M into $\mathbb{P}^1$. For any point $(a_1, a_2) \in \mathbb{P}^1$ consider the section $a_2 f_1 - a_1 f_2 \in \Gamma(M, \mathcal{O}(\zeta_p))$. By Theorem 11 again, there is a unique point $q \in M$ for which this section vanishes, that is, for which $(f_1(q), f_q(q)) = (a_1, a_2) \in \mathbb{P}^1$. The mapping (15) is therefore one-to-one, so that the surface M is analytically equivalent to the surface $\mathbb{P}^1$, and hence the genus must be $g = 0$. This contradicts the hypothesis, and therefore concludes the proof.

Now if M has genus $g > 0$, it follows that the point bundles ζ_p, ζ_q are equal if and only if the points p, q are equal. For if $\zeta_p = \zeta_q$ for $p \neq q$, then the bundle $\zeta_p = \zeta_q$ would have at least two non-trivial sections, one vanishing at p and one at q; and thus $\gamma(\zeta_p) \geq 2$, which is impossible by the preceding Lemma. Therefore, *if* M *has genus* $g > 0$, *the mapping* $p \longrightarrow \zeta_p$ *is a one-to-one mapping from the surface* M *to the subset of* $H^1(M, \mathcal{O}^*)$ *consisting of those complex line bundles* ζ *for which* $c(\zeta) = \gamma(\zeta) = 1$. For the case of genus 0, it follows from the Riemann-Roch theorem that $c(\zeta) = 1$ implies $\gamma(\zeta) = 2$, and thus as in the preceding Lemma, $M = \mathbb{P}^1$. On $\mathbb{P}^1$ there exists a meromorphic function with an arbitrarily prescribed simple zero and simple pole; so any two points p, q are linearly equivalent divisors, and thus $\zeta_p = \zeta_q$ for all points $p, q \in \mathbb{P}^1$. There is hence a unique line bundle ζ on $\mathbb{P}^1$ with $c(\zeta) = 1$. That is, *if* M

has genus $g = 0$, then $M = \mathbb{P}^1$ and all the point bundles of M coincide.

An arbitrary complex line bundle on M can be built up from these point divisors. First, if $\xi \in H^1(M, \mathcal{O}^*)$ is a line bundle for which $\gamma(\xi) > 0$, select a non-trivial section $f \in \Gamma(M, \mathcal{O}(\xi))$ and write $\mathcal{J}(f) = \Sigma_{i=1}^r 1 \cdot p_i$, where $r = c(\xi)$ and the points $p_i \in M$ need not be distinct. It is then clear that $\xi = \zeta_{p_1} \cdots \zeta_{p_r}$; for if $f_i \in \Gamma(M, \mathcal{O}(\zeta_{p_i}))$ are non-trivial sections of the point bundles, so that f_i vanishes precisely at p_i, then $f/f_1 \cdots f_r \in \Gamma(M, \mathcal{O}(\xi\zeta_{p_1}^{-1} \cdots \zeta_{p_r}^{-1}))$ is a holomorphic, nowhere-vanishing section, so necessarily $\xi\zeta_{p_1}^{-1} \cdots \zeta_{p_r}^{-1} = 1$. This representation of course only holds for bundles ξ with $\gamma(\xi) \geq 1$; but if $c(\xi) \geq g$ it follows from (14) that $\gamma(\xi) \geq 1$, and such a representation is possible. Next, for a general line bundle $\xi \in H^1(M, \mathcal{O}^*)$, let $r = c(\xi)$ and select some point $p \in M$ as base point; then $c(\xi \cdot \zeta_p^{g-r}) = g$, so that as above there is a representation $\xi\zeta_p^{g-r} = \zeta_{p_1} \cdots \zeta_{p_r}$. It therefore follows that, having selected a base point $p \in M$, an arbitrary bundle $\xi \in H^1(M, \mathcal{O}^*)$ can be represented in the form

$$\xi = \zeta_{p_1} \cdots \zeta_{p_g} \zeta_p^{r-g} \quad \text{where} \quad r = c(\xi) , \tag{16}$$

the points $p_i \in M$ depending upon the bundle ξ.

The representation (16) may not be unique; if it is not, there will be points $q_1, \ldots, q_g \in M$ such that $\zeta_{p_1} \cdots \zeta_{p_g} \zeta_p^{r-g} = \zeta_{q_1} \cdots \zeta_{q_g} \zeta_p^{r-g}$, hence such that $\zeta_{p_1} \cdots \zeta_{p_g} = \zeta_{q_1} \cdots \zeta_{q_g}$, where the sets $\{p_1, \ldots p_g\} \neq \{q_1, \ldots q_g\}$. This then means that $\gamma(\zeta_{p_1} \cdots \zeta_{p_r}) \geq 2$;

for there is one section with divisor $\Sigma_i 1\cdot p_i$, and another with divisor $\Sigma_i 1\cdot q_i$. Since the converse is clear, it follows that the representation (16) is unique precisely when $\gamma(\zeta_{p_1}\cdots\zeta_{p_g}) = 1$.

To examine this condition further, consider more generally a complex line bundle $\xi = \zeta_{p_1}\cdots\zeta_{p_r}$; the associated divisor $\mathcal{J}_0 = \Sigma_{i=1}^{r} 1\cdot p_i$ can be written $\mathcal{J}_0 = \Sigma_{i=1}^{s} \nu_i\cdot q_i$, where the q_i are the distinct points occurring among the points $p_1,\ldots,p_r$. Let $h_1,\ldots,h_g \in \Gamma(M,\mathcal{O}(\kappa))$ be a basis for the space of Abelian differential forms on the surface M , and as an abbreviation let $\underline{h}$ denote the column vector

$$\underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ \cdots \\ h_g \end{pmatrix}.$$

In terms of a coordinate system z_i centered at the point q_i , the functions h_j are complex analytic functions of the complex variable z_i in an open neighborhood of the origin. The values of these functions and of their derivatives at the origin are well-defined, and will be denoted by $h_j(q_i), h_j'(q_i),\ldots,h_j^{(\nu)}(q_i),\ldots$; and the corresponding column vectors will be denoted by $\underline{h}(q_i),\underline{h}'(q_i),\ldots,\underline{h}^{(\nu)}(q_i),\ldots$. In terms of a different coordinate system centered at the point q_i a different set of values will of course be obtained; the vector $\underline{h}(q_i)$ will be replaced by a nonzero constant multiple of itself, and a vector $\underline{h}^{(\nu)}(q_i)$ will be replaced by a suitable linear combination of the vectors $\underline{h}(q_i)$, $\underline{h}'(q_i),\ldots,\underline{h}^{(\nu)}(q_i)$. Thus the rank

(17) $$\rho = \operatorname{rank}(\underline{h}(q_1),\underline{h}'(q_1),\ldots,\underline{h}^{(\nu_1-1)}(q_1);\ldots;\underline{h}(q_s),\underline{h}'(q_s),\ldots,\underline{h}^{(\nu_s-1)}(q_s))$$

at least will be invariantly defined; the matrix in (17) has $r = \nu_1+\ldots+\nu_s$

Lemma 17. For a complex line bundle $\xi = \zeta_{p_1} \cdots \zeta_{p_r}$ on a compact Riemann surface of genus g,

$$\gamma(\xi) = r - \rho + 1 ,$$

where ρ is the rank of the matrix (17).

Proof. Since $c(\xi) = r$, it follows from the Riemann-Roch theorem that $\gamma(\xi) = r - g + 1 + \gamma(\kappa\xi^{-1})$. If $f_1, \ldots, f_t$ is a basis for the vector space $\Gamma(M, \mathcal{O}(\kappa\xi^{-1}))$, where $t = \gamma(\kappa\xi^{-1})$, and if $g_i \in \Gamma(M, \mathcal{O}(\zeta_{p_i}))$, so that $\mathcal{J}(g_i) = 1 \cdot p_i$, then clearly the elements $f_i g_1 \cdots g_r \in \Gamma(M, \mathcal{O}(\kappa))$, $1 \leq i \leq t$, form a basis for the subspace of Abelian differentials consisting of those elements $h \in \Gamma(M, \mathcal{O}(\kappa))$ such that $\mathcal{J}(h) \geq \mathcal{J}_0$, where $\mathcal{J}_0 = \Sigma_{i=1}^{r} 1 \cdot p_i$. Thus

$$\gamma(\kappa\xi^{-1}) = \dim\{h \in \Gamma(M, \mathcal{O}(\kappa)) \mid \mathcal{J}(h) \geq \mathcal{J}_0\} .$$

Letting $h_1, \ldots h_g$ be a basis for the space of Abelian differentials, any element $h \in \Gamma(M, \mathcal{O}(\kappa))$ can be written uniquely in the form $h = c_1 h_1 + \ldots + c_g h_g$ for some complex constants c_i. The condition that $\mathcal{J}(\varphi) \geq \mathcal{J}_0$ just means that

$$\begin{aligned} h(q_i) &= \Sigma_j\, c_j h_j(q_i) = 0 , \\ h'(q_i) &= \Sigma_j\, c_j h_j'(q_i) = 0 , \\ &\cdots \\ h^{(\nu_i - 1)}(q_i) &= \Sigma_j\, c_j h_j^{(\nu_i - 1)}(q_i) = 0 , \end{aligned}$$

where q_i are the distinct points in the divisor $\mathcal{J}_0$; and this in turn means that the row vector $(c_1, \ldots, c_g)$ is annihilated by the matrix (17). Consequently $\gamma(\kappa\xi^{-1}) = g - \rho$, where ρ is the rank of the matrix (17); so that $\gamma(\xi) = r - g + 1 + \gamma(\kappa\xi^{-1}) = r - \rho + 1$, as desired.

We shall return later to exploit this result more thoroughly, but for the present shall be content with some simple observations. When $r = g$, the matrix appearing in (17) is a $g \times g$ square matrix; the vanishing of its determinant is equivalent to the condition that $\rho < g$, and hence by Lemma 17 is also equivalent to the condition that $\gamma(\xi) > 1$. Therefore $\gamma(\xi) = \gamma(\zeta_{p_1} \cdots \zeta_{p_g}) > 1$ if and only if

$$(18) \quad \det(\underline{h}(q_1), \underline{h}'(q_1), \ldots, \underline{h}^{(\nu_1-1)}(q_1); \ldots; \underline{h}(q_s), \underline{h}'(q_s), \ldots, \underline{h}^{(\nu_s-1)}(q_s)$$

where $\Sigma_{i=1}^{g} 1 \cdot p_i = \Sigma_{i=1}^{s} \nu_i q_i$ and the points q_i are all distinct. In particular, when all the points p_i are distinct, $\gamma(\zeta_{p_1} \cdots \zeta_{p_g}) > 1$ if and only if $\det(\underline{h}(p_1), \underline{h}(p_2), \ldots, \underline{h}(p_g)) = 0$. The functions composing the vector $\underline{\varphi}$ are linearly independent, so that it is evident that this determinant does not vanish identically; there are thus always distinct points p_i so that $\gamma(\zeta_{p_1} \cdots \zeta_{p_g}) = 1$. Indeed, if p_i are distinct points of M and U_i are coordinate neighborhoods about them with coordinate mappings $z_i \colon U_i \longrightarrow \mathbb{C}$, then the function $\det(\underline{h}(z_1)\underline{h}(z_2), \ldots \underline{h}(z_g))$ is a non-trivial complex analytic function of g complex variables in the domain $U_1 \times \ldots \times U_g \subset \mathbb{C}^g$; the set of points at which this determinant vanishes is a proper analytic subvariety of $U_1 \times \ldots \times U_g$, so that in this sense $\gamma(\zeta_{p_1} \cdots \zeta_{p_g}) = 1$ for a general set of g distinct points $p_1, \ldots, p_g$. Thus <u>in general the representation</u> (16) <u>is unique</u>; and in the same sense, as the reader will easily verify, $\gamma(\xi)$ <u>in general takes the minimum value in table</u> (14).

Note in passing that if $h_1, \ldots, h_g \in \Gamma(M, \mathcal{O}(\kappa))$ is a basis for the space of Abelian differentials on a compact Riemann surface M of genus $g > 0$, then <u>the functions</u> h_i <u>have no common zeros</u> on the surface M . For if p is any point of M, $\gamma(\zeta_p) = 1$ by Lemma 16; then applying Lemma 17 in the particular case $r = 1$, it follows that $1 = \gamma(\zeta_p) = 2 - \rho$, hence that $\rho = 1$ where $\rho = \text{rank}(\underline{h}(p))$, so that $h_i(p) \neq 0$ for at least one function h_i.

(d) Of particular interest are divisors of the form $\nu \cdot p$, or equivalently line bundles of the form ζ_p^ν, where $p \in M$ is a given point on the surface and $\nu = 1,2,3,\ldots$. As noted above, there is always an Abelian differential $h_1 \in \Gamma(M, \mathcal{O}(\kappa))$ which is non-zero at the point p, so that $\nu_p(h_1) = 0$. Let $h_2 \in \Gamma(M, \mathcal{O}(\kappa))$ be an Abelian differential which vanishes at p, but such that $\nu_p(h_2) = \rho_2 - 1$ is the minimum possible value. Then let $h_3 \in \Gamma(M, \mathcal{O}(\kappa))$ be an Abelian differential which vanishes et p at least to the order ρ_2, but such that $\nu_p(h_3) = \rho_3 - 1$ is the minimum possible value. Continuing in this manner leads to a basis $h_1,h_2,\ldots,h_g$ for the space of Abelian differentials on the surface M, such that $\nu_p(h_i) = \rho_i - 1$ where

(19) $$1 = \rho_1 < \rho_2 < \ldots < \rho_g < 2g .$$

(Since the total order of an Abelian differential on the surface is $2g - 2$, as noted above in discussing equation (9), it follows that $\nu_p(h_g) = \rho_g - 1 \leq 2g - 2$, hence that $\rho_g < 2g$.) It is clear that this sequence of integers $\rho_i = \rho_i(p)$ is uniquely determined by the point $p \in M$; it is called the Weierstrass gap sequence at the point p, and the values $\rho_i = \rho_i(p)$ are called the Weierstrass gaps at the point p. The significance of this sequence is indicated by the following result.

<u>Theorem 14.</u> Consider the Weierstrass gap sequence at a point p on a compact Riemann surface.

(a) Then

$$\gamma(\zeta_p^\nu) = \nu + 1 - \{\text{number of gaps} \leq \nu\} ;$$

so that

$$\gamma(\zeta_p^\nu) - \gamma(\zeta_p^{\nu-1}) = \begin{cases} 0 & \text{if } \nu \text{ is a gap,} \\ 1 & \text{otherwise.} \end{cases}$$

(b) There exists a meromorphic function on the surface, whose only singularity is a pole of order precisely ν at the point p, if and only if ν is <u>not</u> a gap.

Proof. Let $\underline{h}$ denote the column vector consisting of a basis for the Abelian differentials on the surface, where the basis is normalized as above so that $\nu_p(h_i) = \rho_i - 1$ for the Weierstrass gap sequence $\{\rho_i\}$ at p. Then it is evident that the matrix $(\underline{h}(p), \underline{h}'(p), \underline{h}''(p), \ldots)$ has the form

$$\text{(20)} \qquad \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & \ldots & \rho_2 & \rho_2+1 & \ldots & \rho_3 & \rho_3+1 & \ldots \end{array} \\ \left(\begin{array}{ccccccccc} * & - & \ldots & - & - & \ldots & - & - & \ldots \\ 0 & 0 & \ldots & * & - & \ldots & - & - & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & * & - & \ldots \\ \ldots & \ldots & & \ldots & \ldots & & \ldots & \ldots & \end{array} \right) \end{array} ,$$

where the headings denote the numbers of the columns, and $*$ stands for a non-zero constant. For an integer $\nu \geq 1$, the matrix consisting of the first ν columns of (20) clearly has rank $\rho = \{\text{number of gaps } \rho_i \leq \nu\}$. Therefore from Lemma 17 it follows that

$$\gamma(\zeta_p^\nu) = \nu - \rho + 1 = \nu + 1 - \{\text{number of gaps } \rho_i \leq \nu\} .$$

The second part of assertion (a) is an obvious consequence of this formula.

As for assertion (b), let $g \in \Gamma(M, \mathcal{O}(\zeta_p))$ be any non-trivial section, so that $\mathcal{D}(g) = 1 \cdot p$. Then if $f \in \Gamma(M, \mathcal{O}(\zeta_p^{\nu}))$, the quotient f/g^{ν} will be a meromorphic function on the surface M, whose only singularity is a pole at p of order at most ν; conversely, any such meromorphic function can be so realized. Therefore $\gamma(\zeta_p^{\nu})$ is the dimension of the space of meromorphic functions on M, whose only singularities are poles at p of orders at most ν. If ν is not a gap, then $\gamma(\zeta_p^{\nu}) = \gamma(\zeta_p^{\nu-1}) + 1$, so there must exist a meromorphic function on M whose only singularity is a pole at p, and such that the order is at most ν but not at most $\nu - 1$; in other words, the order is precisely ν. On the other hand, if ν is a gap, then $\gamma(\zeta_p^{\nu}) = \gamma(\zeta_p^{\nu-1})$; so for any function meromorphic on M whose only singularity is a pole at p, if the order of the pole is at most ν, then it must actually be at most $\nu - 1$. This serves to complete the proof.

For any given point $p \in M$ the Weierstrass gaps at the point p satisfy the inequalities $\rho_i(p) \geq i$ for $i = 1,2,\ldots,g$, as a trivial consequence of (19). The discussions in the preceding part (c) might lead one to suspect that in general the Weierstrass gaps would actually take on these minimal values; for by Theorem 14 that would correspond to the dimensions $\gamma(\zeta_p^{\nu})$ being least possible. (Since $\gamma(\zeta_p^{\nu}) \geq 1$, the least possible values for these dimensions differ from the general results given in table (14) in that the zeros of that table are replaced by ones.) This is indeed so, and quite precise results are possible. The <u>Weierstrass weight</u> of a point $p \in M$ is defined in terms of the Weierstrass gaps $\rho_i(p)$ at p by

(21) $$\omega(p) = \sum_{i=1}^{g} (\rho_i(p) - i) .$$

Note that $\omega(p) \geq 0$ for all points $p \in M$. A point $p \in M$ is called a Weierstrass point if $\omega(p) > 0$. The Weierstrass points on the surface M are thus those points at which the Weierstrass gaps do not attain their least possible values. There are actually only a finite number of Weierstrass points on any Riemann surface, as a trivial consequence of the following result.

Theorem 15. If M is a compact Riemann surface of genus g, the Weierstrass weights of the points on the surface M satisfy the equality

$$\sum_{p \in M} \omega(p) = (g-1)\, g(g+1) .$$

Proof. Let $h_1,\dots,h_g \in \Gamma(M, \mathcal{O}(\kappa))$ again be a basis for the space of Abelian differentials on M, and let $\underline{h}$ denote the column vector of length g formed from this basis. If $\{U_\alpha, z_\alpha\}$ is a coordinate covering for the Riemann surface M, then on each set $z_\alpha(U_\alpha) \subset \mathbf{C}$ this vector is a column of complex analytic functions $\underline{h}(z_\alpha)$; and for points in $U_\alpha \cap U_\beta$, where the coordinate transition function is $z_\alpha = f_{\alpha\beta}(z_\beta)$, these column vectors satisfy $\underline{h}(z_\alpha) = \kappa_{\alpha\beta}\, \underline{h}(z_\beta)$ where $\kappa_{\alpha\beta}(z_\beta) = dz_\beta/dz_\alpha$. Differentiating this equation with respect to z_α, it follows that

$$\underline{h}'(z_\alpha) = (dz_\beta/dz_\alpha)\, \frac{d}{dz_\beta}\, (\kappa_{\alpha\beta}(z_\beta)\, \underline{h}(z_\beta)) = \kappa^2_{\alpha\beta}(z_\beta)\, \underline{h}'(z_\beta) + (*)\, \underline{h}(z_\beta) ,$$

where $(*)$ stands for some holomorphic function. Continuing in this manner, in general

$$\underline{h}^{(\nu)}(z_\alpha) = \kappa^{\nu+1}_{\alpha\beta}(z_\beta)\underline{h}^{(\nu)}(z_\beta) + (*)\underline{h}^{(\nu-1)}(z_\beta) + \dots + (*)\underline{h}(z_\beta) . \tag{22}$$

The function

$$g_\alpha(z_\alpha) = \det(\underline{h}(z_\alpha), \underline{h}'(z_\alpha), \dots, \underline{h}^{(g-1)}(z_\alpha))$$

is then holomorphic in $z_\alpha(U_\alpha)$; and from (22) it is evident that in $U_\alpha \cap U_\beta$,

$$\begin{aligned} g_\alpha(z_\alpha) &= \kappa_{\alpha\beta}(z_\beta)^{1+2+\dots+(g-1)} \cdot g_\beta(z_\beta) \\ &= \kappa_{\alpha\beta}(z_\beta)^{g(g-1)/2} \cdot g_\beta(z_\beta) \ . \end{aligned}$$

That is to say, the functions $g = \{g_\alpha(z_\alpha)\}$ define a section of the line bundle $\kappa^{g(g-1)/2}$. Then, by Theorem 11 it follows that

$$(23) \qquad \sum_{p \in M} \nu_p(g) = \frac{g(g-1)}{2} c(\kappa) = (g-1)g(g+1) \ .$$

The order $\nu_p(g)$ is of course unchanged when the functions h_i are subject to any nonsingular linear transformation; so when considering a point $p \in M$ there is no loss of generality in supposing that $\nu_p(h_i) = \rho_i(p) - 1$, where $\{\rho_i(p)\}$ are the Weierstrass gaps at p . If z is a local coordinate mapping defined in a neighborhood of p and such that $z(p) = 0$, then in the power series expansion in terms of the variable z , the function $h_i(z)$ will begin with a term of order precisely $\rho_i - 1$. The lowest order terms in the power series expansion of the function $g(z)$ then obviously come from the expansion

$$\det \begin{pmatrix} z^{\rho_1-1} & (\rho_1-1)z^{\rho_1-2} & \dots & (\rho_1-1)\dots(\rho_1-g+1)z^{\rho_1-g} \\ z^{\rho_2-1} & (\rho_2-1)z^{\rho_2-2} & \dots & (\rho_2-1)\dots(\rho_2-g+1)z^{\rho_2-g} \\ \dots & \dots & & \dots \\ z^{\rho_g-1} & (\rho_g-1)z^{\rho_g-2} & \dots & (\rho_g-1)\dots(\rho_g-g+1)z^{\rho_g-g} \end{pmatrix}.$$

Since the functions z^{ρ_i-1} are linearly independent, this Wronskian determinant cannot vanish identically; and since each monomial in the expansion of the determinant has order

$$(\rho_1+\rho_2+\ldots+\rho_g) - (1+2+\ldots+g) = (\rho_1-1) + (\rho_2-2)+\ldots+(\rho_g-g) = \omega(p) ,$$

that is the order of the full determinant. That is to say, $\nu_p(g) = \omega(p)$; and upon substituting this into equation (23), the desired result follows immediately, thus concluding the proof.

According to Theorem 15, a surface of genus 0 or 1 has no Weierstrass points. This is of course trivial for genus 0 ; while for genus 1 , it could have been noted as a consequence of the last remark in part (c) above, since in that case the unique Abelian differential on the surface is nowhere zero. (As a consequence, the canonical bundle of a surface of genus 1 is necessarily the trivial bundle.) In general, it is clear that the minimum possible weight for a Weierstrass point is $\omega(p) = 1$; and that this corresponds to the Weierstrass gap sequence of the form $1,2,\ldots,g-1,g+1$, where g is the genus of the surface. A Weierstrass point is called a <u>normal Weierstrass point</u> if it has this minimal form, that is, if $\omega(p) = 1$; and the surface M is called a <u>normal Riemann surface</u> if all of its Weierstrass points are normal Weierstrass points. By Theorem 15, a normal Riemann surface has precisely $(g-1)g(g+1)$ Weierstrass points. By Theorem 14, the dimensions $\gamma(\zeta_p^\nu)$ can be read off immediately, and depend merely upon whether or not p is one of the Weierstrass points; and the meromorphic functions having but a single pole on the surface must have a pole or order at least g .

The consideration of the other extreme behavior of Weierstrass points is a bit more subtle, and depends on the following observation. For a fixed point p on the surface, suppose that

ν_1, ν_2 are non-gap values; then by Theorem 14 there are meromorphic functions f_1, f_2 on the surface, such that f_i has as its sole singularity a pole of order precisely ν_i at the point p. The product $f_1 f_2$ has a pole of order precisely $\nu_1 + \nu_2$ at p, and is regular otherwise; so by Theorem 14 again, $\nu_1 + \nu_2$ is also a non-gap value. Therefore, <u>the set of non-gap values at a point is closed under addition</u>, (forms an additive sub-semigroup of the positive integers). Letting r be the least non-gap value at the point p, it follows that whenever $\nu > r$ is a gap value, then $\nu - r$ is also a gap; consequently, all the gaps occur in finite arithmetical sequences of the form $i, i+r, i+2r, \dots, i + \lambda_i r$, (where $i = 1, 2, \dots, r-1$ and $\lambda_i = 0, 1, 2, \dots$). In particular, a point p is called a <u>hyperelliptic Weierstrass point</u> if its least non-gap value is 2; at such a point the Weierstrass gap sequence has the form $1, 3, 5, \dots, 2g-1$. The weight of a hyperelliptic Weierstrass point is

$$\begin{aligned} \omega(p) &= [1 + 3 + \dots + (2g-1)] - [1 + 2 + \dots + g] \\ &= [1 + 2 + \dots + 2g] - 3[1 + 2 + \dots + g] \\ &= \tfrac{1}{2} g (g-1) . \end{aligned}$$

A Riemann surface is called a <u>hyperelliptic surface</u> if all of its Weierstrass points are hyperelliptic Weierstrass points; by Theorem 15, such a surface will have $2(g+1)$ Weierstrass points altogether. By Theorem 14, the dimensions $\gamma(\zeta_p^\nu)$ can be read off immediately, and depend merely upon whether or not p is one of the Weierstrass points; and for each hyperelliptic Weierstrass point there exists a meromorphic function having a double pole at that point but being regular otherwise. (A Riemann surface of genus $g = 1$ is called an <u>elliptic</u> Riemann surface. For such a surface, it follows from the Riemann-Roch theorem that $\gamma(\zeta_p^2) = 2$ for every point p; and therefore, for every point p on an elliptic surface there exists a meromorphic function having a double pole at that point but being regular otherwise. For this reason, elliptic curves are sometimes considered as falling within the class of hyperelliptic curves, even though they have no Weierstrass points at all. However we shall not adopt this convention, but shall distinguish between elliptic and hyperelliptic surfaces; so hyperelliptic surfaces all have genus $g > 1$.)

Now turning to the case of a general Weierstrass point, the following assertion can be made.

<u>Theorem 16.</u> If p is a Weierstrass point on a compact Riemann surface of genus g, its weight satisfies the inequality

$$1 \leq \omega(p) \leq \frac{1}{2} g(g-1) ;$$

$\omega(p) = 1$ precisely when p is a normal Weierstrass point, and $\omega(p) = \frac{1}{2} g(g-1)$ precisely when p is a hyperelliptic Weierstrass point. The total number N of Weierstrass points on the surface satisfies the inequality

$$2(g+1) \leq N \leq (g-1)\, g(g+1) ;$$

$N = 2(g+1)$ precisely when the surface is hyperelliptic, and $N = (g-1)\, g(g+1)$ precisely when the surface is normal.

Proof. Letting $r > 1$ be the least non-gap value at a Weierstrass point p, the gap values can be written in the form $i,\ i+r,\ i+2r, \ldots, i+\lambda_i r$, $(i = 1,2,\ldots,r-1;\ \lambda_i = 0,1,2,\ldots)$.

The total number of gaps is

$$g = \sum_{i=1}^{r-1} (\lambda_i + 1) = r - 1 + \sum_{i=1}^{r-1} \lambda_i \ . \tag{24}$$

The weight of the point p is then

$$\omega(p) = \sum_{i=1}^{r-1} \sum_{j=0}^{\lambda_i} (i + jr) - \frac{1}{2} g(g+1)$$

$$= \sum_{i=1}^{r-1} [i(\lambda_i + 1) + \frac{1}{2} r \lambda_i(\lambda_i + 1)] - \frac{1}{2} g(g+1)$$

$$= \frac{1}{2} r(r-1) + \sum_{i=1}^{r-1} \frac{1}{2} \lambda_i (2i + r\lambda_i + r) - \frac{1}{2} g(g+1) .$$

Since $(i + r\lambda_i)$ is a gap value, it follows from the inequality (19) that $(i + r\lambda_i) \leq 2g - 1$; so $2i + r\lambda_i + r \leq i + (2g-1) + r \leq (r-1) + (2g-1) + r = 2(g+r-1)$. Then from (24),

$$\sum_{i=1}^{r-1} \frac{1}{2} \lambda_i (2i + r\lambda_i + r) \leq (g + r - 1) \sum_{i=1}^{r-1} \lambda_i = (g+r-1)(g-r+1) .$$

Therefore

$$\omega(p) \leq \frac{1}{2} g(g-1) - \frac{1}{2}(r-1)(r-2) .$$

The maximum value of $\omega(p)$ is therefore $\frac{1}{2} g(g-1)$, and this is attained only for $r = 2$, which is the hyperelliptic case. The assertion about the minimum value for $\omega(p)$ was proved earlier. Finally, the results about the total number of Weierstrass points follow immediately from the inequalities on the weights and Theorem 15, thus concluding the proof.

For emphasis, recall again that surfaces of genus 0 or 1 have no Weierstrass points.

§8. Picard and Jacobi varieties

(a) The first step in the detailed classification of complex line bundles over a compact Riemann surface M was taken in §7(a), with the introduction of the Chern class of a line bundle; from the exact sequence of sheaves

$$(1) \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0$$

there followed the exact cohomology sequence

$$(2) \qquad 0 \longrightarrow H^1(M, \mathcal{O})/H^1(M,\mathbb{Z}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{c} \mathbb{Z} \longrightarrow 0 ,$$

where c is the homomorphism associating to a complex line bundle its Chern class. The subgroup of complex line bundles having Chern class zero is thus isomorphic to the group $H^1(M, \mathcal{O})/H^1(M,\mathbb{Z})$, and the investigation of this group is the next step. To begin, consider the following exact sequence of sheaves

$$(3) \qquad 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \mathcal{O}^{1,0} \longrightarrow 0 ,$$

where d is the operation of exterior differentiation. The associated exact cohomology sequence over the Riemann surface M begins

$$0 \longrightarrow \Gamma(M,\mathbb{C}) \longrightarrow \Gamma(M, \mathcal{O}) \longrightarrow \Gamma(M, \mathcal{O}^{1,0}) \xrightarrow{\delta} H^1(M,\mathbb{C}) \longrightarrow$$
$$\longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^{1,0}) \longrightarrow H^2(M,\mathbb{C}) \longrightarrow H^2(M, \mathcal{O}) \longrightarrow \dots .$$

Since M is compact, $\Gamma(M,\mathbb{C}) = \Gamma(M, \mathcal{O}) = \mathbb{C}$, and also $H^2(M,\mathbb{C}) = \mathbb{C}$. Furthermore, $H^2(M, \mathcal{O}) = 0$ by Theorem 8, and $H^1(M, \mathcal{O}^{1,0}) \cong H^0(M, \mathcal{O}) = \mathbb{C}$ by the Serre duality theorem. Therefore the above exact cohomology sequence leads to the exact sequence

$$(4) \qquad 0 \longrightarrow \Gamma(M, \mathcal{O}^{1,0}) \xrightarrow{\delta} H^1(M,\mathbf{C}) \longrightarrow H^1(M, \mathcal{O}) \longrightarrow 0 .$$

Since the inclusion $\mathbf{Z} \subset \mathcal{O}$ can be factored through the inclusions $\mathbf{Z} \subset \mathbf{C} \subset \mathcal{O}$, it follows that the homomorphism $H^1(M,\mathbf{Z}) \longrightarrow H^1(M, \mathcal{O})$ in the exact sequence (1) can be factored through the homomorphism $H^1(M,\mathbf{C}) \longrightarrow H^1(M, \mathcal{O})$ in the exact sequence (4). Consequently

$$(5) \qquad \frac{H^1(M, \mathcal{O})}{H^1(M,\mathbf{Z})} \cong \frac{H^1(M,\mathbf{C})}{H^1(M,\mathbf{Z}) + \delta\Gamma(M, \mathcal{O}^{1,0})} ,$$

where δ is the coboundary homomorphism arising from the exact sequence (3). For later purposes, recall that this homomorphism has the following explicit form. For any element $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$, and for any suitable coordinate covering $\mathcal{U} = \{U_\alpha\}$ of M , there will exist holomorphic functions $f_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ such that $\varphi = df_\alpha$ in U_α . The constants

$$(\delta\varphi)_{\alpha\beta} = f_\beta - f_\alpha$$

form a one-cocycle $(\delta\varphi)_{\alpha\beta} \in Z^1(\mathcal{U}, \mathbf{C})$, representing the cohomology class $\delta\varphi \in H^1(M,\mathbf{C})$. Note further that this homomorphism is actually an injection; indeed, the following somewhat stronger assertion can be made.

<u>Lemma 18.</u> Consider the homomorphism $\delta: \Gamma(M, \mathcal{O}^{1,0}) \to H^1(M,\mathbf{C})$ for a compact Riemann surface M . If $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$ is an element such that

$$\delta\varphi \in H^1(M,\mathbf{R}) \subset H^1(M,\mathbf{C}) ,$$

then $\varphi = 0$.

Proof. In terms of the above explicit form for the homomorphism δ, it is apparent that $\delta\varphi \in H^1(M,\mathbb{R})$ if and only if, for a suitable coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the surface M, there exist holomorphic functions $f_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ such that $\varphi = df_\alpha$ in U_α and such that $f_\beta - f_\alpha$ is real. The functions $g_\alpha = \exp 2\pi i\, f_\alpha$ are thus holomorphic in U_α, and $|g_\alpha| = |g_\beta|$ in $U_\alpha \cap U_\beta$. Since M is compact, the globally defined function $|g_\alpha|$ must attain its maximum at some point of M; but then g_α is constant in an open neighborhood of that point by the maximum modulus theorem, and hence all the functions g_α are constant by the identity theorem for analytic functions. The functions f_α are necessarily constant as well, so $\varphi = df_\alpha = 0$, as asserted.

Another, although equivalent, approach to the classification can be made through a slightly different exact sequence of sheaves. For any germ of function $f \in \mathcal{O}^*$ define

$$d\ell(f) = \frac{1}{2\pi i}\, d \log f \in \mathcal{O}^{1,0} ;$$

this is clearly a sheaf homomorphism $d\ell : \mathcal{O}^* \longrightarrow \mathcal{O}^{1,0}$. The mapping $d\ell$ is onto; for any germ of holomorphic differential form $\varphi \in \mathcal{O}^{1,0}$ can be written $\varphi = dg$ for some function $g \in \mathcal{O}$, and then $\varphi = d\ell(\exp 2\pi i\, g)$. The kernel of $d\ell$ is the subsheaf $\mathbb{C}^* \subset \mathcal{O}^*$ of constant functions. There is thus the exect sequence of sheaves

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{O}^* \xrightarrow{d\ell} \mathcal{O}^{1,0} \longrightarrow 0 .$$

The associated exact cohomology sequence over the Riemann surface M begins

$$\longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{c^*} H^1(M, \mathcal{O}^{1,0}) \longrightarrow H^2(M,\mathbf{C}^*) \longrightarrow H^2(M, \mathcal{O}^*) \rightarrow \dots .$$

Since M is compact, $\Gamma(M,\mathbf{C}^*) = \Gamma(M, \mathcal{O}^*) = \mathbf{C}^*$, and also $H^2(M,\mathbf{C}^*) = \mathbf{C}^*$. Furthermore, $H^2(M, \mathcal{O}^*) = 0$; for from the exact cohomology sequence associated to (1) there is the segment $H^2(M, \mathcal{O}) \longrightarrow H^2(M, \mathcal{O}^*) \longrightarrow H^3(M,\mathbf{Z})$, while $H^2(M, \mathcal{O}) = 0$ by Theorem 8, and $H^3(M,\mathbf{Z}) = 0$ since dimension $(M) = 2$. The Serre duality theorem shows that $H^1(M, \mathcal{O}^{1,0}) \cong H^0(M, \mathcal{O}) = \mathbf{C}$; and the image of the mapping c^* is hence the kernel of a homomorphism $\mathbf{C} \longrightarrow \mathbf{C}^*$, and will be denoted by $\tilde{\mathbf{Z}}$. (Note that $\tilde{\mathbf{Z}}$ is the kernel of a particular homomorphism $\mathbf{C} \longrightarrow \mathbf{C}^*$; as might be expected, we shall shortly identify $\mathbf{Z}$ with $\tilde{\mathbf{Z}}$.) As a consequence of these observations, the above exact cohomology sequence leads to the exact sequence

$$(6) \qquad 0 \longrightarrow \Gamma(M, \mathcal{O}^{1,0}) \xrightarrow{\delta^*} H^1(M,\mathbf{C}^*) \xrightarrow{i^*} H^1(M, \mathcal{O}^*) \xrightarrow{c^*} \tilde{\mathbf{Z}} \longrightarrow 0 .$$

The group $H^1(M,\mathbf{C}^*)$ will be called the group of <u>flat complex line bundles</u> over M. The homomorphism i^* is that induced by the natural inclusion mapping $\mathbf{C}^* \longrightarrow \mathcal{O}^*$; so the image of i^* thus consists of those complex line bundles ξ which admit representative cocycles $(\xi_{\alpha\beta})$ consisting of constant functions.

The Serre duality theorem merely asserts that $H^1(M, \mathcal{O}^{1,0})$ is canonically dual to the group $H^0(M, \mathcal{O}) = \mathbf{C}$; the actual isomorphism $H^1(M, \mathcal{O}^{1,0}) \cong \mathbf{C}$ therefore involves an element of choice, which will be made in the following manner. Considering a cohomology class $\chi \in H^1(M, \mathcal{O}^{1,0})$ as a linear functional on $\mathbf{C}$, associate to that cohomology class the complex constant $\chi(-1)$. With this choice, the homomorphism c^* in the exact sequence (6) can be described as follows.

Lemma 19. For a complex line bundle $\xi \in H^1(M, \mathcal{O}^*)$ over a compact Riemann surface M , $c^*(\xi) \in \mathbf{C}$ is the Chern class of that bundle.

Proof. This is a straightforward matter of tracing through the various mappings involved. First, the explicit form of the duality in Serre's theorem was described in §5(b). For any cohomology class $\sigma \in H^1(M, \mathcal{O}^{1,0})$ select a representative cocycle $(\sigma_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^{1,0})$ in terms of a suitable locally finite coordinate covering $\mathcal{U} = \{U_\alpha\}$ of M . There is a zero cochain $(\tau_\alpha) \in C^0(\mathcal{U}, \mathcal{E}^{1,0})$ having coboundary $(\sigma_{\alpha\beta})$, so $\sigma_{\alpha\beta} = \tau_\beta - \tau_\alpha$ in $U_\alpha \cap U_\beta$. Then $\bar{\partial}\tau_\beta = \bar{\partial}\tau_\alpha$, and the differential form $(\bar{\partial}\tau_\alpha) \in \Gamma(M, \mathcal{E}^{1,1})$ represents the cohomology class σ under the Dolbeault isomorphism $H^1(M, \mathcal{O}^{1,0}) \cong \Gamma(M, \mathcal{E}^{1,1})/\bar{\partial}\Gamma(M, \mathcal{E}^{0,1})$. Thus the constant corresponding to the cohomology class σ under our chosen isomorphism $H^1(M, \mathcal{O}^{1,0}) \cong \mathbf{C}$ is just $-\iint_M(\bar{\partial}\tau_\alpha)$. Next, for a complex line bundle $\xi \in H^1(M, \mathcal{O}^*)$ with a representative cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$, the image $\sigma = c^*(\xi)$ is represented by the cocycle $\sigma_{\alpha\beta} = d\ell(\xi_{\alpha\beta}) = \frac{1}{2\pi i} d \log \xi_{\alpha\beta}$. As in Lemma 14, select nowhere-vanishing C^∞ functions r_α in the various sets U_α so that $r_\beta = r_\alpha|\xi_{\alpha\beta}|^2$ in the $U_\alpha \cap U_\beta$; then

$$\sigma_{\alpha\beta} = \frac{1}{2\pi i}\, \partial \log \xi_{\alpha\beta} = \frac{1}{2\pi i}\, (\partial \log r_\beta - \partial \log r_\alpha) .$$

Thus in the explicit form of the Serre duality mapping we can put

$$c^*(\xi) = - \iint_M \frac{1}{2\pi i} \bar{\partial}\partial(\log r_\alpha) = \frac{1}{2\pi i} \iint_M \partial\bar{\partial} \log(r_\alpha) \ .$$

However, by Lemma 14 the latter integral is precisely the Chern class $c(\xi)$, concluding the proof.

It follows immediately from Lemma 19 and the exact sequence (6) that the complex line bundles arising from flat line bundles are precisely those complex line bundles of Chern class zero; or equivalently, the necessary and sufficient condition that a complex line bundle $\xi \in H^1(M, \mathcal{O}^*)$ admit a representative cocycle $(\xi_{\alpha\beta})$ consisting entirely of constant functions is that $c(\xi) = 0$.

In summary of the preceding, the group of complex line bundles of Chern class zero can be described in the following three equivalent forms:

$$(7) \quad \{\xi \in H^1(M, \mathcal{O}^*) \,|\, c(\xi) = 0\} \cong \begin{cases} \dfrac{H^1(M, \mathbf{C})}{H^1(M, \mathbf{Z}) + \delta\Gamma(M, \mathcal{O}^{1,0})} \\[2ex] \dfrac{H^1(M, \mathcal{O})}{H^1(M, \mathbf{Z})} \\[2ex] \dfrac{H^1(M, \mathbf{C}^*)}{\delta^*\Gamma(M, \mathcal{O}^{1,0})} \end{cases} .$$

Moreover, the isomorphisms (7) lead to isomorphisms between the three groups on the right-hand side, which are explicitly as follows. First, from the exact cohomology sequence associated to the exact sheaf sequence (4) there arises the homomorphism $H^1(M, \mathbf{C}) \longrightarrow H^1(M, \mathcal{O})$, which induces an isomorphism

$$\frac{H^1(M, \mathbf{C})}{H^1(M, \mathbf{Z}) + \delta\Gamma(M, \mathcal{O}^{1,0})} \longrightarrow \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbf{Z})} \ .$$

Next, from the exact cohomology sequence associated to the exact sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{e} \mathbb{C}^* \longrightarrow 0$$

there arises the homomorphism $H^1(M,\mathbb{C}) \longrightarrow H^1(M,\mathbb{C}^*)$, which induces an isomorphism

$$\frac{H^1(M,\mathbb{C})}{H^1(M,\mathbb{Z}) + \delta\Gamma(M,\mathcal{O}^{1,0})} \longrightarrow \frac{H^1(M,\mathbb{C}^*)}{\delta^*\Gamma(M,\mathcal{O}^{1,0})} .$$

The first assertion was proved above; verification of the second assertion will be left to the reader.

(b) The expressions (7) permit an additional structure to be imposed on the group of complex line bundles of Chern class zero. In general, a lattice subgroup of a finite dimensional real vector space is defined to be an additive subgroup of the vector space generated by a set of elements which are linearly independent over the real numbers; and a lattice subgroup of a finite dimensional complex vector space is defined to be a lattice subgroup of the associated real vector space. Thus in an n dimensional complex vector space, with its natural associated structure as a $2n$ dimensional real vector space, a lattice subgroup has at most $2n$ generators. Considering the first form given in (7), recall that $H^1(M,\mathbb{C})$ has the natural structure of a $2g$ dimensional complex vector space, where g is the genus of the Riemann surface M. As is known (cf. the topological appendix), $H^1(M,\mathbb{Z}) \subset H^1(M,\mathbb{C})$ is a lattice subgroup; indeed, any $2g$ generators of $H^1(M,\mathbb{Z})$ form a basis for the complex vector space $H^1(M,\mathbb{C})$. By Lemma 18 the homomorphism $\delta: \Gamma(M, \mathcal{O}^{1,0}) \longrightarrow H^1(M,\mathbf{C})$ is injective, hence the image is a g dimensional complex linear subspace; so the quotient space

$$V = H^1(M,\mathbf{C})/\delta\Gamma(M, \mathcal{O}^{1,0})$$

has the structure of a g dimensional complex vector space. Let $\lambda_1,\ldots,\lambda_{2g}$ be a set of generators for the lattice subgroup $H^1(M,\mathbf{Z}) \subset H^1(M,\mathbf{C})$; and let $\tilde{\lambda}_1,\ldots,\tilde{\lambda}_{2g}$ be the corresponding elements in the quotient space V. Then $\tilde{\lambda}_1,\ldots,\tilde{\lambda}_{2g}$ generate a lattice subgroup of V. For if there were real numbers $x_1,\ldots,x_{2g}$, not all zero, such that $\Sigma_i\, x_i\tilde{\lambda}_i = 0$, then $\Sigma_i\, x_i\lambda_i \in \delta\Gamma(M, \mathcal{O}^{1,0}) \subset H^1(M,\mathbf{C})$; but then by Lemma 18 this would imply that $\Sigma_i\, x_i\lambda_i = 0$, contradicting the fact that $\lambda_1,\ldots,\lambda_{2g}$ generate a lattice subgroup. As a consequence, the quotient group

$$\frac{H^1(M,\mathbf{C})}{H^1(M,\mathbf{Z}) + \delta\Gamma(M, \mathcal{O}^{1,0})}$$

has the structure of the quotient space of a g dimensional complex vector space V by a lattice subgroup generated by $2g$ elements; this structure will be called the <u>Picard variety</u> of the Riemann surface M , and will be denoted by $\underset{\sim}{P}(M)$. As remarked above, $V = H^1(M,\mathbf{C})/\delta\Gamma(M, \mathcal{O}^{1,0}) \cong H^1(M, \mathcal{O})$, so the same structure can be described in the form $\underset{\sim}{P}(M) = H^1(M, \mathcal{O})/H^1(M,\mathbf{Z})$. The Picard variety is in particular an Abelian group, and its role in the classification of complex line bundles lies in its occurrence in the exact sequence

$$(8) \qquad 0 \longrightarrow \underset{\sim}{P}(M) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{c} \mathbf{Z} \longrightarrow 0 .$$

The group of all complex line bundles over M therefore has the natural structure $H^1(M, \mathcal{O}^*) \cong \mathbf{Z} + \underset{\sim}{P}(M)$.

To consider in more detail this additional structure, let V be any g dimensional complex vector space, and $\mathcal{L} \subset V$ be a lattice subgroup of V. Of course, as an abstract group $\mathcal{L} \cong \mathbf{Z}^r$ for some integer $r \leq 2g$, which will be called the <u>rank</u> of the lattice subgroup; the rank clearly can be characterized also as the dimension of the real vector subspace of V spanned by the elements of $\mathcal{L}$. We shall consider here only the case in which $\mathcal{L}$ has the maximal possible rank. First, ignoring the complex structure, consider V as a $2g$ dimensional real vector space. A set of generators of $\mathcal{L}$ can be used for a basis for the real vector space V, so that $V/\mathcal{L} \cong \mathbf{R}^{2g}/\mathbf{Z}^{2g} = (\mathbf{R}/\mathbf{Z})^{2g}$. The space $V/\mathcal{L}$ can thus be factored as a Cartesian product of $2g$ circles, and thereby has the structure of a compact manifold of dimension $2g$. The vector space V is obviously the universal covering space of this manifold, the covering mapping being the natural projection $V \longrightarrow V/\mathcal{L}$; thus $\pi_1(V/\mathcal{L}) \cong \mathbf{Z}^{2g}$. Now returning to the complex structure, V can be considered as a complex analytic manifold, and the covering mapping $V \longrightarrow V/\mathcal{L}$ defines a natural complex analytic structure on the quotient space $V/\mathcal{L}$; those coordinate neighborhoods on V small enough to project homeomorphically to $V/\mathcal{L}$ can be taken as coordinate neighborhoods on $V/\mathcal{L}$. Therefore $V/\mathcal{L}$ has the structure of a compact complex analytic manifold; a manifold of this form will be called a <u>complex analytic torus</u>. It is obvious that the group operations are complex analytic, in the sense that the mapping

$$V/\mathcal{L} \times V/\mathcal{L} \longrightarrow V/\mathcal{L}$$

defined by $(p,q) \longrightarrow p-q$ is a complex analytic mapping. Thus $V/\mathcal{L}$ is a complex Lie group, indeed an Abelian complex Lie group. (Lie groups here are connected.)

As those who are familiar with Lie groups know, any compact Abelian complex Lie group is of the form $V/\mathcal{L}$. (See for instance, C. Chevalley, Theory of Lie Groups I, (Princeton, 1946); the discussion there is for the real case, but goes through in the same manner in the complex case.) It is obvious that an isomorphism $V/\mathcal{L} \longrightarrow V'/\mathcal{L}'$ between two compact Abelian complex Lie groups is equivalent to a complex linear isomorphism $V \longrightarrow V'$ which takes the lattice $\mathcal{L}$ into the lattice $\mathcal{L}'$. Therefore the structure we are investigating is nothing more nor less than that of a compact Abelian complex Lie group. Actually, however, the Lie group aspect need not be considered any further; for the complex structure itself essentially carries all the information.

<u>Lemma 20.</u> Let V, V' be g dimensional complex vector spaces, and $\mathcal{L} \subset V$, $\mathcal{L}' \subset V'$ be lattice subgroups of rank $2g$. The compact complex manifolds $V/\mathcal{L}$, $V'/\mathcal{L}'$ are holomorphically equivalent if and only if there is a complex linear isomorphism $F\colon V \longrightarrow V'$ such that $F(\mathcal{L}) = \mathcal{L}'$.

Proof. First, a complex linear isomorphism $F\colon V \longrightarrow V'$ is a complex analytic homeomorphism; and if $F(\mathcal{L}) = \mathcal{L}'$ it is evident that F induces a complex analytic homeomorphism $f\colon V/\mathcal{L} \longrightarrow V'/\mathcal{L}'$. Next, assume conversely that there is a complex analytic homeomorphism $f\colon V/\mathcal{L} \longrightarrow V'/\mathcal{L}'$. The composition of the natural projection $V \longrightarrow V/\mathcal{L}$ and the mapping f

yields an analytic local homeomorphism $V \longrightarrow V'/\mathcal{L}'$; and since V and V' are simply connected, the latter mapping can be factored through an analytic local homeomorphism $F: V \longrightarrow V'$. That is to say, there will exist a complex analytic local homeomorphism $F: V \longrightarrow V'$ which induces the given mapping $f: V/\mathcal{L} \longrightarrow V'/\mathcal{L}'$; so that for any element $\lambda \in \mathcal{L}$ there will exist an element $\lambda' \in \mathcal{L}'$ such that

$$F(p+\lambda) = F(p) + \lambda' \tag{9}$$

for all points $p \in V$. In terms of coordinate systems $(z_1,\ldots,z_g)$ for V and $(w_1,\ldots,w_g)$ for V' , the mapping F will be given by a g-tuple $w_i = F_i(p)$ of complex analytic functions of g complex variables. Differentiating equation (9), it follows that

$$(\partial F_i/\partial z_j)(p+\lambda) = (\partial F_i/\partial z_j)(p)$$

for all points $p \in V$ and all elements $\lambda \in \mathcal{L}$. The functions $\partial F_i/\partial z_j$ are thus invariant under $\mathcal{L}$, and so define complex analytic functions on $V/\mathcal{L}$; but since $V/\mathcal{L}$ is compact, it follows from the maximum modulus theorem as in Lemma 1 that $\partial F_i/\partial z_j$ is constant. The mapping F is consequently linear; and being a local homeomorphism, F is nonsingular. For F to induce a homeomorphism from $V/\mathcal{L}$ onto $V/\mathcal{L}'$, it is then necessary that $F(\mathcal{L}) = \mathcal{L}'$, and the proof is therewith concluded.

It is then evident from this Lemma that two compact Abelian complex Lie groups $V/\mathcal{L}$ and $V'/\mathcal{L}'$ are isomorphic precisely when their underlying complex analytic manifolds are analytically equivalent. Therefore in future investigations the group structure can be ignored, in part. For emphasis, it should be repeated that a complex analytic manifold will only be called a complex analytic torus when it is analytically equivalent to a manifold of the form $V/\mathcal{L}$, where V is a g dimensional complex vector space and $\mathcal{L} \subset V$ is a lattice subgroup of rank $2g$; and thus a complex analytic torus carries a unique further structure of a compact Abelian complex Lie group.

It is frequently useful to be much more explicit in the description of a lattice subgroup or complex torus. So choose a basis for the vector space V, or equivalently, an isomorphism $V \cong \mathbb{C}^g$; the elements of $\mathbb{C}^g$ will be written as complex column vectors of length g. Also choose a set of generators $\lambda_1, \ldots, \lambda_{2g}$ for the lattice subgroup $\mathcal{L}$. Each vector $\lambda_j \in \mathcal{L} \subset \mathbb{C}^g$ will be represented by a column vector $\lambda_j = (\lambda_{ij})$, $i = 1, \ldots, g$; and the set of all $2g$ of these vectors form a $g \times 2g$ matrix $\Lambda = (\lambda_{ij})$ called a <u>period matrix</u> for the lattice subgroup $\mathcal{L} \subset V$ or complex torus $V/\mathcal{L}$. Then the $2g \times 2g$ square matrix

$$\begin{pmatrix} \Lambda \\ \bar{\Lambda} \end{pmatrix},$$

where $\bar{\Lambda}$ denotes the complex conjugate of the matrix Λ, will be called an associated full period matrix.

<u>Lemma 21.</u> A complex matrix Λ of g rows and $2g$ columns is a period matrix for a compact complex torus if and only if its associated full period matrix $\begin{pmatrix} \Lambda \\ \bar{\Lambda} \end{pmatrix}$ is nonsingular.

Proof. By definition, the columns of Λ generate a lattice of maximal rank in $\mathbb{C}^g$ (equivalently define a complex torus) if and only if they are linearly independent over the real numbers; therefore the contradiction of this situation is the assertion that there exists a real column vector x of length $2g$, not identically zero, such that $\Lambda x = 0$. If there exists such a vector x, then $\begin{pmatrix}\Lambda \\ \overline{\Lambda}\end{pmatrix} x = 0$, so the full period matrix is singular. Conversely suppose that the full period matrix is singular; there will then exist a complex column vector $z = x + iy$ of length $2g$, not identically zero, such that $\begin{pmatrix}\Lambda \\ \overline{\Lambda}\end{pmatrix} z = 0$, or equivalently, such that $\Lambda z = \Lambda \overline{z} = 0$. But then $\Lambda x = \Lambda y = 0$, where x,y are real and not both are identically zero, and that suffices to complete the proof.

Remark. There are of course several equivalent ways of expressing the condition that a $g \times 2g$ complex matrix Λ be the period matrix of a compact complex torus; the version used in the preceding lemma is perhaps the simplest to state. The version closest to the definition is that, for a real column vector $x \in \mathbb{R}^{2g}$, $\Lambda x = 0$ if and only if $x = 0$; this is just the assertion that the columns of Λ are linearly independent over the real numbers, and the preceding lemma demonstrated that this assertion is equivalent to the assertion that the full period matrix be non-singular. Yet another version is that for a complex column vector $z \in \mathbb{C}^g$, ${}^t z\Lambda$ is a real vector if and only if $z = 0$; here ${}^t z$ denotes the transpose of z, hence a row vector. To show that this assertion is equivalent to the condition of the lemma, it suffices to show that there exists a non-zero vector $z \in \mathbb{C}^g$ such that ${}^t z\Lambda$ is real if and only if $\begin{pmatrix}\Lambda \\ \overline{\Lambda}\end{pmatrix}$ is singular. First suppose that ${}^t z\Lambda$ is real, that is, that ${}^t z\Lambda = {}^t \overline{z\Lambda}$, for a non-zero vector z; then

$$({}^t z, -{}^t\bar{z}) \begin{pmatrix} \Lambda \\ \bar{\Lambda} \end{pmatrix} = {}^t z\Lambda - \overline{{}^t z\Lambda} = 0 ,$$

hence $\begin{pmatrix} \Lambda \\ \bar{\Lambda} \end{pmatrix}$ is singular. Conversely, suppose that $\begin{pmatrix} \Lambda \\ \bar{\Lambda} \end{pmatrix}$ is singular; writing $\Lambda = R + iS$ where R, S are real matrices, this is equivalent to the condition that $\binom{R}{S}$ is singular. Since this is a real matrix, there must exist a non-zero vector $\binom{a}{b} \in \mathbb{R}^{2g}$ so that $({}^t a, {}^t b)\binom{R}{S} = {}^t aR + {}^t bS = 0$; but then

$$\operatorname{Im} \{({}^t b + i\,{}^t a)\Lambda\} = \operatorname{Im} \{({}^t b + i\,{}^t a)(R + iS)\} = {}^t aR + {}^t bS = 0 ,$$

which proves the desired assertion.

The period matrix associated to a complex torus is not unique, since two arbitrary choices were made; so it is important to examine the effects of these choices. First, choosing a different basis for the vector space V amounts to applying an isomorphism $\mathbb{C}^g \cong \mathbb{C}^g$; representing this isomorphism by a non-singular $g \times g$ complex matrix M , the period matrix is obviously transformed into the period matrix $M\Lambda$, since each vector λ_j is transformed into the vector $M\lambda_j$. Second, a different basis $\lambda'_1, \ldots, \lambda'_{2g}$ for the lattice subgroup $\mathcal{L}$ is necessarily of the form $\lambda'_j = \Sigma_k \lambda_k n_{kj}$, where $N = (n_{kj})$ is a $2g \times 2g$ matrix of integer elements and of determinant ± 1 ; this change clearly replaces the period matrix Λ by the matrix ΛN . Consequently, <u>two period matrices</u> Λ, Λ' <u>represent the same complex torus if and only if</u>

(10) $$\Lambda' = M\Lambda N \quad \text{where} \quad M \in GL(g,\mathbb{C}),\ N \in GL(2g,\mathbb{Z}) .$$

(Here $GL(n,R)$ denotes the group of invertible $n \times n$ matrices over the ring R .) This equivalence relation can be used to bring a period matrix into a simpler form, as follows. Decompose a period matrix Λ into two square $g \times g$ blocks of the form $\Lambda = (\Lambda_1, \Lambda_2)$. By multiplying Λ on the right by a suitable integer matrix $N \in GL(2g, \mathbb{Z})$, it can be arranged that the matrix Λ_2 be non-singular; to see this, merely recall that there are g linearly independent columns in Λ, and a suitable matrix N can be found rearranging the columns to make these the last g columns. Then the matrix Λ and the matrix $\Lambda_2^{-1}\Lambda = (\Lambda_2^{-1}\Lambda_1, I)$ represent the same complex torus; so <u>any complex torus can be represented by a period matrix of the form</u> $\Lambda = (\Lambda_1, I)$, where I denotes the $g \times g$ identity matrix. This is still far from associating a unique period matrix to a complex torus, however. To proceed further, suppose that $\Lambda = (\Lambda_1, I)$ and $\Lambda' = (\Lambda_1', I)$ represent the same complex torus, so that $\Lambda' = M\Lambda N$ as in (10). Decompose N into the $g \times g$ matrix blocks

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ;$$

then

$$(\Lambda_1', I) = M(\Lambda_1, I)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (M(\Lambda_1 A + C),\ M(\Lambda_1 B + D)) .$$

Hence $\Lambda_1 B + D$ is non-singular, and $M = (\Lambda_1 B + D)^{-1}$; and $\Lambda_1' = (\Lambda_1 B + D)^{-1}(\Lambda_1 A + C)$. The converse being apparent, it follows that <u>two period matrices matrices</u> $\Lambda = (\Lambda_1, I)$ <u>and</u> $\Lambda' = (\Lambda_1', I)$ <u>represent the same complex torus if and only if there exists a matrix</u>

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2g, \mathbb{Z})$$

<u>such that</u> $\Lambda_1 B + D$ <u>is non-singular and</u>

(11) $$\Lambda_1' = (\Lambda_1 B + D)^{-1}(\Lambda_1 A + C) .$$

Note that, by Lemma 21, a matrix (Λ_1, I) is the period matrix of a complex torus if and only if $\operatorname{Im} \Lambda_1$ is non-singular, where Im denotes the imaginary part of the matrix.

It is useless to proceed any further in this direction just at the present point; but the special case $g = 1$ provides an interesting and illustrative example. The period matrix can be taken in the form $\Lambda = (\lambda_1, 1)$, where $\lambda_1 \in \mathbb{C}$ and $\operatorname{Im} \lambda_1 \neq 0$; and two matrices $\Lambda = (\lambda_1, 1)$ and $\Lambda' = (\lambda_1', 1)$ represent the same complex torus if and only if there is a matrix

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

such that

$$\lambda_1' = \frac{a\lambda_1 + c}{b\lambda_1 + d} \quad ;$$

note that $b\lambda_1 + d$ is always non-zero, since $\operatorname{Im} \lambda_1 \neq 0$. This is in fact precisely the equivalence relation discussed in §1(f), for the complex tori are the compact Riemann surfaces of genus 1 discussed there. Note that it is evident that there actually are distinct complex analytic tori, so that the structure introduced is a non-trivial one.

Finally, to obtain an explicit description of the Picard variety $\underset{\sim}{P}(M)$ of a compact Riemann surface M of genus g, choose a basis for $H^1(M, \mathbb{Z})$, or equivalently, an isomorphism $H^1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}$; the same elements form a basis for $H^1(M, \mathbb{C})$, so there is a corresponding isomorphism $H^1(M, \mathbb{C}) \cong \mathbb{C}^{2g}$. Further, choose a basis $\varphi_1, \ldots, \varphi_g \in \Gamma(M, \mathcal{O}^{1,0})$ for the space of Abelian differentials on the Riemann surface M; the associated cohomology classes will then be of the form

$$\delta\varphi_i = \begin{pmatrix} \omega_{1\ i} \\ \omega_{2\ i} \\ \cdots \\ \omega_{2g\ i} \end{pmatrix} \in \mathbb{C}^{2g} .$$

The collection of all such vectors form a $2g \times g$ matrix $\Omega = (\omega_{ji})$ called the period matrix of the Abelian differentials on M. Note that by Lemma 18, if $z \in \mathbb{C}^g$ is a vector such that $\Omega z \in \mathbb{R}^{2g}$, then necessarily $z = 0$; hence by Lemma 21 (recalling the remarks following that lemma), the matrix Ω is the period matrix of a compact complex torus. (Actually, to parallel the earlier discussion we should consider the transposed matrix ${}^t\Omega$ rather than Ω; but this merely amounts to considering the rows of Ω rather than the columns of ${}^t\Omega$, a trivial distinction which will be left to the reader to sort out.) The period matrix of the Abelian differentials is of course not unique, but depends on the choices of bases for $H^1(M,\mathbb{Z})$ and for $\Gamma(M, \mathcal{O}^{1,0})$; it is quite obvious that different choices have the effect of replacing the matrix Ω by a matrix $N\Omega M$, where $N \in GL(2g,\mathbb{Z})$ and $M \in GL(g,\mathbb{C})$. Therefore, recalling equation (10), all of these choices lead to the same compact complex torus, which will be called the Jacobi variety $\underset{\sim}{J}(M)$ of the Riemann surface M.

Now the Picard variety of M is given by

$$\underset{\sim}{P}(M) = \frac{H^1(M,\mathbb{C})}{H^1(M,\mathbb{Z}) + \delta\Gamma(M, \mathcal{O}^{1,0})} ;$$

so selecting any linear mapping $\Lambda\colon H^1(M,\mathbb{C}) \longrightarrow \mathbb{C}^g$ having precisely of $(M, \mathcal{O}^{*})$ as kernel, $\underset{\sim}{P}(M)$ is the compact complex torus $\underset{\sim}{P}(M) = \mathbb{C}^g/\Lambda H^1(M,\mathbb{Z})$. In terms of the isomorphism $H^1(M,\mathbb{C}) \cong \mathbb{C}^{2g}$ chosen above, the mapping $\Lambda: \mathbb{C}^{2g} \longrightarrow \mathbb{C}^g$ is represented by a $g \times 2g$ complex matrix $\Lambda = (\lambda_{ij})$; since the image of Λ must be all of $\mathbb{C}^g$, the matrix Λ must have have rank g . The condition that $\Lambda(\delta\Gamma(M, \mathcal{O}^{1,0})) = 0$ is evidently that $\Lambda\Omega = 0$, since the columns of Ω span the subspace $\delta\Gamma(M, \mathcal{O}^{1,0}) \subset \mathbb{C}^{2g}$. The lattice subgroup $H^1(M,\mathbb{Z}) = \mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ is generated by the $2g$ column vectors e_i , where the entries of e_i are zero except for a one in the i-th place; and since $\Lambda \cdot e_i$ is just the i-th column of the matrix Λ , the image $\Lambda H^1(M,\mathbb{Z}) \subset \mathbb{C}^g$ is the lattice generated by the $2g$ column vectors of the matrix Λ . Therefore Λ itself is a period matrix of the Picard variety of M .

In summary then, let Ω be a period matrix for the Abelian differentials on M ; the Jacobi variety $\underset{\sim}{J}(M)$ of M is the compact complex torus defined by the period matrix ${}^t\Omega$. Let Λ be any $g \times 2g$ matrix of rank g such that $\Lambda\Omega = 0$; the Picard variety $\underset{\sim}{P}(M)$ of M is the compact complex torus defined by the period matrix Λ .

(c) Not every compact complex torus can be the Jacobi or Picard variety of a Riemann surface; in fact, it is still an unsolved problem to describe precisely which tori arise from Riemann surfaces. A very important partial answer is provided in the form of an additional structure which the Jacobi and Picard varieties inherit from the multiplicative structure of the cohomology of a surface.

Recall that there is a skew-symmetric bilinear mapping

$$H^1(M,\mathbb{C}) \times H^1(M,\mathbb{C}) \longrightarrow H^2(M,\mathbb{C})$$

on the cohomology of a surface M, called the cup product. Perhaps the easiest way to describe this is in terms of differential forms. Under the deRham isomorphism as described in §5(a),

$$H^1(M,\mathbb{C}) \cong \frac{\{\varphi \in \Gamma(M, \mathcal{E}^1) | d\varphi = 0\}}{d\Gamma(M, \mathcal{E}^0)} ;$$

so any cohomology class in $H^1(M,\mathbb{C})$ is represented by a closed differential form. If φ,ψ are two such forms, their product $\varphi \wedge \psi$ is a closed differential form of degree 2, hence represents an element of $H^2(M,\mathbb{C})$ under the deRham isomorphism. It is clear that the cohomology class represented by $\varphi \wedge \psi$ is unchanged when either φ or ψ is replaced by a differential form representing the same one-dimensional cohomology class; for instance, if $f \in \Gamma(M, \mathcal{E}^0)$, then $(\varphi + df) \wedge \psi = \varphi \wedge \psi + d(f\psi)$. The mapping

$$(\varphi,\psi) \longrightarrow \varphi \wedge \psi$$

then defines the cup product operation in cohomology. To vary the description slightly, consider the natural identification $H^2(M,\mathbb{C}) \cong \mathbb{C}$ as introduced in §5(a). The cup product can then be envisaged as a bilinear mapping $H^1(M,\mathbb{C}) \times H^1(M,\mathbb{C}) \longrightarrow \mathbb{C}$; and in terms of differential forms, this mapping can be described as

$$(\varphi,\psi) \longrightarrow \iint_M \varphi \wedge \psi .$$

Note that the subgroup $H^1(M,\mathbb{R}) \times H^1(M,\mathbb{R}) \subset H^1(M,\mathbb{C}) \times H^1(M,\mathbb{C})$ is mapped into the subgroup $H^2(M,\mathbb{R}) = \mathbb{R}$ under the cup product, and

. . . -- mapped into the subgroup

$H^2(M,\mathbf{Z}) = \mathbf{Z}$. The cup product of cohomology classes $\alpha,\beta \in H^1(M,\mathbf{C})$ will be denoted by $\alpha \cup \beta$, considered as an element of $\mathbf{C}$.

Choosing a basis for the group $H^1(M,\mathbf{Z})$, that is, an isomorphism $H^1(M,\mathbf{Z}) \cong \mathbf{Z}^{2g}$, the cup product $\mathbf{Z}^{2g} \times \mathbf{Z}^{2g} \longrightarrow \mathbf{Z}$ is defined by a skew-symmetric $2g \times 2g$ integer matrix X , called the <u>intersection matrix</u> of the surface M ; explicitly, if $m,n \in \mathbf{Z}^{2g}$ are column vectors representing one-dimensional cohomology classes, their cup product is the integer ${}^t m X n$. The same matrix of course describes the cup product in real or complex cohomology, in terms of the same basis. A change of basis in $H^1(M,\mathbf{Z})$ is described by a matrix $N \in GL(2g,\mathbf{Z})$; and this replaces the intersection matrix X by the intersection matrix ${}^t N^{-1} X N^{-1}$.

<u>Theorem 17.</u> Let M be a compact Riemann surface of genus $g > 0$; and let X be the intersection matrix and Ω be the period matrix of the Abelian differentials on M ; in terms of some basis for $H^1(M,\mathbf{Z})$. Then

(i) ${}^t\Omega X \Omega = 0$, (Riemann's equality); and

(ii) $i\,{}^t\Omega X \Omega$ is positive definite Hermitian, (Riemann's inequality).

Proof. Let $\varphi_1,\dots,\varphi_g \in \Gamma(M, \mathcal{O}^{1,0})$ be a basis for the Abelian differentials on M , so that the cohomology classes $\delta\varphi_i = (\omega_{ji}) \in H^1(M,\mathbf{C}) = \mathbf{C}^{2g}$ form the column vectors of the matrix Ω . Note that the differential forms φ_i are closed, and that the mapping $\varphi_i \longrightarrow \delta\varphi_i$ of (4) coincides with the deRham mapping;

the conjugate differentials $\overline{\varphi}_i$ are also closed, and under the deRham mapping correspond to the conjugate cohomology classes. Thus the cup products of these cohomology classes can be calculated by integrating products of the Abelian differentials and their conjugates. Firstly, note that $\varphi_i \wedge \varphi_j \equiv 0$, since the product would be a form of bidegree $(2,0)$; therefore the cup product $\iint_M \varphi_i \wedge \varphi_j = 0$. In terms of the intersection matrix X, however, this cup product is given by

$$0 = \sum_{k,\ell} \omega_{ki} X_{k\ell} \omega_{\ell j} = ({}^t\Omega X \Omega)_{ij} ,$$

that is to say, is the entry in row i, column j, of the matrix ${}^t\Omega X \Omega$; and hence ${}^t\Omega X \Omega = 0$. Secondly, if $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$ is any Abelian differential, then in a coordinate neighborhood U_α with a coordinate mapping z_α that differential form can be written $\varphi = h_\alpha(z_\varphi) dz_\alpha$, where $h_\alpha(z_\alpha)$ is a holomorphic function; thus

$$i\varphi \wedge \overline{\varphi} = i|h_\alpha(z_\alpha)|^2 dz_\alpha \wedge d\overline{z}_\alpha = 2|h_\alpha(z_\alpha)|^2 dx_\alpha \wedge dy_\alpha ,$$

writing $z_\alpha = x_\alpha + iy_\alpha$. Since $dx_\alpha \wedge dy_\alpha$ is the local area element, then for the cup product it follows that

$$i\iint_M \varphi \wedge \overline{\varphi} \geqq 0 ,$$

with equality occurring only when $\varphi = 0$. In terms of the basic Abelian differentials, write $\varphi = \Sigma_i c_i \varphi_i$ for some complex constants c_i; so

$$0 \leqq \sqrt{-1} \iint_M \varphi \wedge \overline{\varphi} = \sum_{i,j} c_i \overline{c}_j \sqrt{-1} \iint_M \varphi_i \wedge \overline{\varphi}_j = \sum_{i,j} c_i P_{ij} \overline{c}_j ,$$

where

$$P_{ij} = \sqrt{-1} \iint_M \varphi_i \wedge \overline{\varphi}_j .$$

The matrix $P = (P_{ij})$ is clearly Hermitian, and the above inequality shows that it is positive definite as well. In terms of the intersection matrix however,

$$P_{ij} = \sqrt{-1} \sum_{k,\ell} \omega_{ki} X_{k\ell} \bar{\omega}_{\ell j} = \sqrt{-1} \, ({}^t\Omega X \bar{\Omega})_{ij} \ ;$$

or in other words, $P = i\,{}^t\Omega X \bar{\Omega}$, which serves to conclude the proof.

A matrix Ω with $2g$ rows and g columns is called a Riemann matrix if there exists a skew-symmetric integer matrix X with $2g$ rows and columns, such that:

$$(12) \quad \begin{cases} \text{(i)} & {}^t\Omega X \Omega = 0 , \quad \text{and} \\ \text{(ii)} & i\,{}^t\Omega X \bar{\Omega} \ \text{is positive definite Hermitian.} \end{cases}$$

The matrix X is called a principal matrix for the Riemann matrix Ω ; the set (Ω, X) consisting of a Riemann matrix Ω and an associated principal matrix is called a Riemann matrix pair, and it is this pair which is of primary interest. Theorem 17 is then the assertion that the set (Ω, X) , consisting of the period matrix Ω of the Abelian differentials on a compact Riemann surface and the intersection matrix of the surface (in terms of some choice of basis for $H^1(M, \mathbb{Z})$), is a Riemann matrix pair.

If (Ω, X) is a Riemann matrix pair, and if $M \in GL(g, \mathbb{C})$ and $N \in GL(2g, \mathbb{Z})$ are arbitrary matrices, then clearly $(N\Omega M, {}^tN^{-1} X N^{-1})$ is also a Riemann matrix pair; this merely amounts to verifying condition (12), a trivial calculation which will be left to the reader. Two Riemann matrix pairs (Ω, X) and (Ω', X') will be called equivalent when $\Omega' = N\Omega M$ and $X' = {}^tN^{-1} X N^{-1}$ for some matrices $M \in GL(g, \mathbb{C})$ and $N \in GL(2g, \mathbb{Z})$. Recalling equation

(10), it is evident that the period matrices ${}^t\Omega$ and ${}^t\Omega'$ represent the same complex torus whenever (Ω, X) and (Ω', X') are equivalent Riemann matrix pairs. A complex torus whose period matrix is a Riemann matrix will be called an <u>Abelian variety</u>; and the preceding observation shows that all the period matrices representing an Abelian variety are necessarily Riemann matrices. Actually of course, an equivalence class of Riemann matrix pairs determines an Abelian variety with an additional structure; such an equivalence class will be called a <u>polarized Abelian variety</u>, and will be viewed as an Abelian variety with the additional structure determined by a fixed principal matrix for any period matrix defining the underlying complex torus. With this terminology, Theorem 17 has the following immediate consequence.

<u>Corollary.</u> The Jacobi variety of a compact Riemann surface of genus $g > 0$ has a canonical structure as a polarized Abelian variety.

Proof. For any choice of a basis for $H^1(M,\mathbb{Z})$ and a basis for $\Gamma(M, \mathcal{O}^{1,0})$, Theorem 17 shows that the pair (Ω, X), consisting of the period matrix of that basis for the Abelian differentials and the intersection matrix for the surface, is a Riemann matrix pair. Another choice of a basis for $H^1(M,\mathbb{Z})$ is determined by any matrix $N \in GL(2g,\mathbb{Z})$; and another choice for a basis for $\Gamma(M, \mathcal{O}^{1,0})$ is determined by any matrix $M \in GL(g,\mathbb{C})$. As noted earlier, the new period matrix is then $N\Omega M$, and the new intersection matrix is ${}^tN^{-1}XN^{-1}$; so this leads to an equivalent Riemann matrix pair, sufficing to conclude the proof.

The discussion so far has concerned only the Jacobi variety $\underset{\sim}{J}(M)$ of the Riemann surface M, and it remains to show how this structure is reflected in the Picard variety $\underset{\sim}{P}(M)$. Again choosing bases for $H^1(M,\mathbb{Z})$ and for $\Gamma(M, \mathcal{O}^{1,0})$, let X be the intersection matrix of the surface and Ω be the period matrix of the Abelian differentials on the surface; and let Λ be any $g \times 2g$ matrix of rank g such that $\Lambda\Omega = 0$.

Lemma 22. The Riemann matrix pairs (Ω, X) and $({}^t\Lambda, {}^tX^{-1})$ are canonically equivalent.

Proof. That (Ω, X) is a Riemann matrix pair was proved in Theorem 17. To show that $({}^t\Lambda, {}^tX^{-1})$ is an equivalent Riemann matrix pair it is only necessary to find matrices $M \in GL(g,\mathbb{C})$ and $N \in GL(2g,\mathbb{Z})$ such that ${}^t\Lambda = N\Omega M$ and ${}^tX^{-1} = {}^tN^{-1}XN^{-1}$; and selecting the matrices M,N in a canonical manner will show that the equivalence is canonical. Let P be the non-singular $g \times g$ matrix $P = i\,{}^t\Omega X \overline{\Omega}$, and let Q be the $g \times g$ matrix $Q = \Lambda\overline{\Omega}$; since $\Lambda\Omega = 0$, rank $\Lambda = g$, and rank $(\Omega,\overline{\Omega}) = 2g$, it is clear that the matrix Q is also non-singular. The proof will be completed by verifying that the matrices $M = -i\,{}^tP^{-1}\,{}^tQ$ and $N = X$ satisfy the desired conditions. It is obvious that ${}^tN^{-1}XN^{-1} = {}^tX^{-1}XX^{-1} = {}^tX^{-1}$. For the remaining condition, note that

$$\begin{pmatrix} \overline{\Lambda} \\ \Lambda \end{pmatrix} (\Omega,\overline{\Omega}) = \begin{pmatrix} \overline{\Lambda}\Omega & \overline{\Lambda}\overline{\Omega} \\ \Lambda\Omega & \Lambda\overline{\Omega} \end{pmatrix} = \begin{pmatrix} \overline{Q} & 0 \\ 0 & Q \end{pmatrix} ;$$

and therefore

$$\begin{pmatrix} \overline{Q}^{-1} & \overline{\Lambda} \\ Q^{-1} & \Lambda \end{pmatrix} = (\Omega,\overline{\Omega})^{-1} .$$

Consequently

$$ {}^t M {}^t \Omega {}^t N = - {}^t M {}^t \Omega X = - {}^t M {}^t \Omega X(\Omega,\overline{\Omega}) \begin{pmatrix} \overline{Q}^{-1} & \overline{\Lambda} \\ Q^{-1} & \Lambda \end{pmatrix} $$

$$ = {}^t M (0, iP) \begin{pmatrix} \overline{Q}^{-1} & \overline{\Lambda} \\ Q^{-1} & \Lambda \end{pmatrix} = {}^t M i P Q^{-1} \Lambda = \Lambda , $$

so $N\Omega M = {}^t\Lambda$, as desired.

This lemma shows that <u>the Picard variety $\underset{\sim}{P}(M)$ and the Jacobi variety $\underset{\sim}{J}(M)$ of a compact Riemann surface are canonically isomorphic</u> as compact complex tori, hence as compact complex Lie groups; and thus that the Picard variety also has a canonical structure as a polarized Abelian variety. It is convenient to retain both of these names, with their connotations of different simple standard representations of the same Abelian variety $\underset{\sim}{P}(M) = \underset{\sim}{J}(M)$; and it is important for later purposes to note the explicit form of the isomorphism between these two representations. Considering $\underset{\sim}{J}(M)$ as the quotient of the vector space $\mathbb{C}^g$ of column vectors (z) by the lattice generated by the columns of ${}^t\Omega$, and considering $\underset{\sim}{P}(M)$ as the quotient of the vector space $\mathbb{C}^g$ of column vectors (w) by the lattice generated by the columns of Λ , write

$$ \underset{\sim}{J}(M) = \frac{\mathbb{C}^g : (z)}{{}^t\Omega \mathbb{Z}^{2g}} , \qquad \underset{\sim}{P}(M) = \frac{\mathbb{C}^g : (w)}{\Lambda\, \mathbb{Z}^{2g}} . $$

The isomorphism is then given by

(13) $$ w = - iQP^{-1} z . $$

The verification is straightforward, and will be left to the reader.

that, for a suitable choice of basis for $H^1(M,\mathbb{Z})$, the intersection matrix of the surface has the form

$$(14)\qquad X = J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I denotes the $g \times g$ identity matrix and 0 the $g \times g$ zero matrix; a basis of this form will be called a <u>canonical basis</u> for the surface. (The notation J or J_g will be reserved for matrices of the form (14).) A Riemann matrix pair defining the (polarized) Jacobi variety of a compact Riemann surface can thus always be taken in the canonical form (Ω,J). Splitting the matrix Ω into $g \times g$ matrix blocks of the form $\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$, the Riemann conditions take the form

$$(15)\quad \begin{cases} \text{(i)} & {}^t\Omega_2\Omega_1 - {}^t\Omega_1\Omega_2 = 0\text{, and} \\ \text{(ii)} & i({}^t\Omega_2\overline{\Omega}_1 - {}^t\Omega_1\overline{\Omega}_2) \text{ is positive definite Hermitian;} \end{cases}$$

this is an easy calculation, which will be left to the reader. <u>Thus $\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$ is a Riemann matrix with associated principal matrix J if and only if (15) holds.</u> Note that the matrices Ω_1 and Ω_2 are both necessarily non-singular. For if Ω_2 were singular, there would exist a non-zero constant vector $c \in \mathbb{C}^g$ such that $\Omega_2 c = 0$; but then ${}^t c\,{}^t\Omega_2 = \overline{\Omega}_2\overline{c} = 0$, hence $i\,{}^t c({}^t\Omega_2\overline{\Omega}_1 - {}^t\Omega_1\overline{\Omega}_2)\overline{c} = 0$, contradicting the positive definiteness of the matrix (15ii).

As in the earlier discussion of the period matrices of general compact complex tori, so in the discussion of Riemann matrix pairs, it is of some interest to examine simpler representatives of of an equivalence class. If J is the associated principal matrix of one representative of the equivalence class, then we can restrict consideration to those Riemann matrix pairs of the form (Ω, J) in the given equivalence class; and since the principal matrix is fixed, it suffices merely to consider the Riemann matrix Ω alone. The most general Riemann matrix Ω' in the same (restricted) equivalence class is then of the form $\Omega' = N \Omega M$, where $N \in GL(2g, \mathbb{Z})$, $M \in GL(g, \mathbb{C})$, and ${}^tN^{-1} J N^{-1} = J$ (or equivalently, $J = {}^tNJN$). The set of all matrices $N \in GL(2g, \mathbb{Z})$ such that ${}^tNJN = J$ form a group called the <u>symplectic group</u> of rank g, and denoted by $SP(g, \mathbb{Z})$. Writing $\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$, the matrix Ω_2 is non-singular, as noted above; hence $\Omega' = \Omega\Omega_2^{-1}$ is an equivalent Riemann matrix. Thus, <u>an equivalence class of Riemann matrix pairs with principal matrix</u> J <u>always contains a Riemann matrix of the form</u> $\Omega = \begin{pmatrix} \Omega_1 \\ I \end{pmatrix}$. In this case, it is readily seen that equations (15) take the form

$$(16) \quad \begin{cases} \text{(i)} & \Omega_1 \text{ is symmetric, and} \\ \text{(ii)} & \operatorname{Im} \Omega_1 \text{ is positive definite.} \end{cases}$$

<u>Thus</u> $\Omega = \begin{pmatrix} \Omega_1 \\ I \end{pmatrix}$ <u>is a Riemann matrix with associated principal matrix</u> J <u>if and only if</u> Ω_1 <u>satisfies (16).</u> Now suppose that $\Omega = \begin{pmatrix} \Omega_1 \\ I \end{pmatrix}$ and $\Omega' = \begin{pmatrix} \Omega_1' \\ I \end{pmatrix}$ are two equivalent such matrices; there must then exist matrices $N \in SP(g, \mathbb{Z})$ and $M \in GL(g, \mathbb{Z})$ such that $\Omega' = N \Omega M$. And writing $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, it follows as in equation (11) that

$$(16') \qquad \Omega_1' = (A\Omega_1 + B)(C\Omega_1 + D)^{-1} ;$$

thus <u>two Riemann matrices</u> $\begin{pmatrix} \Omega_1 \\ I \end{pmatrix}, \begin{pmatrix} \Omega_1' \\ I \end{pmatrix}$ <u>with associated principal matrix</u> J <u>are equivalent if and only if there exists a sympletic</u>

$\begin{pmatrix} \\ C & D \end{pmatrix}$ such that (15') holds. Note that, as a consequence of the obvious observation in the preceding paragraph, $C\Omega_1 + D$ is always non-singular. The set of matrices Ω_1 satisfying (15) is called the Siegel generalized upper half space, and denoted by $\mathfrak{S}_g$. The symplectic group $SP(g,\mathbf{Z})$ acts as a group of transformations on $\mathfrak{S}_g$, and the quotient space $\mathfrak{S}_g/SP(g,\mathbf{Z})$ is in one-to-one correspondence with the equivalence classes of Riemann matrix pairs (Ω,J). This leads to a fascinating area of study, but there is not enough time to continue further in this direction here. Let it suffice merely to remark that the symplectic group acts discontinuously, and that the quotient space is a complex analytic space having the same dimension as $\mathfrak{S}_g$, namely $\frac{1}{2}(g+1)$. Note that it is clear from this discussion that not all complex tori are Abelian varieties.

(d) It naturally occurs to one to enquire about the relationships between the complex structure of a Riemann surface M, and the complex structure of the Picard or Jacobi variety of M. To pose a precise question, for any two points $p,q \in M$ consider the associated point bundles ζ_p, ζ_q; the complex line bundle $\xi = \zeta_p\zeta_q^{-1}$ has Chern class zero, hence can be considered as an element of the Picard variety $\underset{\sim}{P}(M)$. There is thus a mapping $M \times M \longrightarrow \underset{\sim}{P}(M)$, defined by $(p,q) \longrightarrow \zeta_p\zeta_q^{-1} \in \underset{\sim}{P}(M)$; and the question is whether this is a complex analytic mapping.

Lemma 23. On a compact Riemann surface M of genus $g > 0$ consider any two points p,q, and the complex line bundle

$\xi = \zeta_p \zeta_q^{-1} \in \underset{\sim}{P}(M)$; and let τ be an arbitrary differentiable arc from the point q to the point p in M. Then representing the Picard variety in the form

$$\underset{\sim}{P}(M) = \frac{H^1(M, \mathbf{C})}{H^1(M, \mathbf{Z}) + \delta\Gamma(M, \mathcal{O}^{1,0})},$$

the line bundle $\xi \in \underset{\sim}{P}(M)$ can be represented by a cohomology class $\sigma \in H^1(M, \mathbf{C})$ such that

(17) $$\sigma \cup (\delta\varphi) = \int_\tau \varphi$$

for every Abelian differential $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$.

Proof. To begin, suppose that the points p, q, and the arc τ lie in a contractible coordinate neighborhood $U_0 \subset M$; the coordinate mapping z_0 can be taken to be a homeomorphism of an open neighborhood of the point set closure $\overline{U}_0 \subset M$ into the complex line $\mathbf{C}$, mapping $\overline{U}_0$ to a closed disc. Choose an open set $V_0 \subset M$ such that $\tau \subset V_0 \subset \overline{V}_0 \subset U_0$. Further, choose an open covering of M consisting of the set U_0 and some additional sets $\{U_\alpha\}$, with the properties that: (i) $U_\alpha \cap V_0 = \emptyset$ for $\alpha \neq 0$; and (ii) the covering $\mathfrak{U}$ is a Leray covering for the sheaves $\mathcal{O}$ and $\mathbf{C}$. (Recalling §3(e) and §3(f), it is only necessary to require that the sets U_α be coordinate neighborhoods, and that these sets and all their intersections be contractible.) Since $c(\xi) = 0$, the line bundle ξ can be represented by a cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathbf{C}^*)$ formed of constant functions. And since $\xi = \zeta_p \zeta_q^{-1}$, there will be a meromorphic section $(f_\alpha) \in \Gamma(\mathcal{M}^*(\xi))$ with divisor $\mathcal{J}(f_\alpha) = 1\cdot p - 1\cdot q$; the functions f_α thus satisfy $f_\alpha = \xi_{\alpha\beta} f_\beta$ in $U_\alpha \cap U_\beta$. In each set U_α for $\alpha \neq 0$ the function f_α is holomorphic and nowhere vanishing; selecting some branch of the logarithm, introduce the holomorphic functions

$$\sigma_\alpha = -\frac{1}{2\pi i}\log f_\alpha \in \Gamma(U_\alpha, \mathcal{O}) .$$

In the annular region $U_0 - V_0$ the function $\log f_0$ will also be single-valued, since the sum of the residues of the function $d \log f_0$ within V_0 is zero; so select a C^∞ function σ_0 in the set U_0 which coincides with $-\frac{1}{2\pi i}\log f_0$ in the annular region $U_0 - V_0$. (For instance, select a branch of $-\frac{1}{2\pi i}\log f_0$ in an open neighborhood of the point set closure $\overline{U}_0 - V_0$, and multiply it by a C^∞ function which is identically one on $U_0 - V_0$ and which vanishes identically on a suitable subset of V_0.) Then defining $\sigma_{\alpha\beta} = \sigma_\beta - \sigma_\alpha$ in $U_\alpha \cap U_\beta$, it follows that $\exp 2\pi i\, \sigma_{\alpha\beta} = f_\alpha/f_\beta = \xi_{\alpha\beta}$; these functions are therefore constants, so form a cocycle $(\sigma_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathbb{C})$. Recalling §8(a), this cocycle determines a cohomology class $\sigma \in H^1(M, \mathbb{C})$ which represents the complex line bundle ξ.

To calculate the cup product $\sigma \cup (\delta\varphi)$ of this cohomology class with the cohomology class $\delta\varphi \in H^1(M, \mathbb{C})$ represented by an Abelian differential $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$, observe that the differential form $(d\sigma_\alpha) \in \Gamma(M, \mathcal{E}^1)$ represents the cohomology class σ under the deRham isomorphism; and obviously, φ represents the cohomology class $\delta\varphi$ under the deRham isomorphism. Consequently $\sigma \cup \delta\varphi = \iint_M d\sigma_\alpha \wedge \varphi$. Since both φ and $d\sigma_\alpha$ are holomorphic differential forms, hence differential forms of degree $(1,0)$, everywhere outside of the set U_0, it follows that $d\sigma_\alpha \wedge \varphi \equiv 0$ on $M - U_0$; so that $\sigma \cup \delta\varphi = \iint_{U_0} d\sigma_0 \wedge \varphi$. Select a holomorphic function h in the set U_0 so that $dh = \varphi$; then from Stokes' theorem,

$$\sigma \cup \delta\varphi = \iint_{U_0} d\sigma_0 \wedge dh = - \iint_{U_0} d(h d\sigma_0)$$
$$= -\int_{\partial U_0} h d\sigma_0 \quad .$$

The function σ_0 was chosen so that it coincided with $-\frac{1}{2\pi i}\log f_0$ along the boundary ∂U_0 ; so applying the residue theorem, and remembering the definition of the function h ,

$$\sigma \cup \delta\varphi = -\int_{\partial U_0} h\, d\sigma_0 = \frac{1}{2\pi i}\int_{\partial U_0} h\, d\log f_0$$
$$= h(p) - h(q) = \int_\tau dh = \int_\tau \varphi \quad .$$

This demonstrates the theorem in the special case that $p, q,$ and τ lie within a coordinate neighborhood. In general, any arc τ can be split into segments τ_i from points q_i to q_{i+1} , such that q_i, q_{i+1}, and τ_i lie in a coordinate neighborhood; there will then be a cohomology class $\sigma_i \in H^1(M,\mathbb{C})$ representing the line bundle $\zeta_{q_{i+1}}\zeta_{q_i}^{-1}$, so that $\sigma_i \cup \delta\varphi = \int_{\tau_i}\varphi$. The sum $\sigma = \Sigma_i\sigma_i \in H^1(M,\mathbb{C})$ represents the line bundle $\Pi_i\zeta_{q_{i+1}}\zeta_{q_i}^{-1} = \zeta_p\zeta_q^{-1} = \xi$, and

$$\sigma \cup \delta\varphi = \Sigma_i(\sigma_i \cup \delta\varphi) = \Sigma_i \int_{\tau_i}\varphi = \int_\tau \varphi \ ,$$

which completes the proof in general.

Now choose a basis for $H^1(M,\mathbb{Z})$, thus identifying $H^1(M,\mathbb{C}) = \mathbb{C}^{2g}$, $H^1(M,\mathbb{Z}) = \mathbb{Z}^{2g}$; and choose a basis $\varphi_1,\ldots,\varphi_g \in \Gamma(M, \mathcal{O}^{1,0})$ for the Abelian differentials on M. Let X be the intersection matrix of the surface M in this basis,

Ω be the period matrix of the Abelian differentials on M, and Λ be any $g \times 2g$ matrix of rank g such that $\Lambda\Omega = 0$; and introduce the non-singular $g \times g$ matrices

$$P = i\,{}^t\Omega X \bar{\Omega}\,, \qquad Q = \Lambda\bar{\Omega}\,.$$

The Picard-Jacobi variety of the surface M then has the standard representations

$$\underset{\sim}{J}(M) = \frac{\mathbb{C}^g}{{}^t\Omega\mathbb{Z}^{2g}}\,, \qquad \underset{\sim}{P}(M) = \frac{\mathbb{C}^g}{\Lambda\mathbb{Z}^{2g}}\,,$$

which are related as in equation (13). With this notation, and the convention that all vectors are viewed as column vectors, the following holds.

Theorem 18. (Abel's Theorem) Let τ be any differentiable arc from a point q to a point p on the compact Riemann surface M, and introduce the vector $t \in \mathbb{C}^g$ with coordinates

$$t_i = \int_\tau \varphi_i\,, \qquad i = 1,\dots,g\,.$$

Then the complex line bundle $\xi = \zeta_p\zeta_q^{-1} \in H^1(M, \mathcal{O}^*)$, considered as an element of the Picard-Jacobi variety $\underset{\sim}{J}(M) = \underset{\sim}{P}(M)$ of the surface M, is given explicitly as

$$\text{(18)} \qquad \xi = \begin{cases} t \in \mathbb{C}^g/{}^t\Omega\mathbb{Z}^{2g} = \underset{\sim}{J}(M)\,, \\ -\,i\,QP^{-1}t \in \mathbb{C}^g/\Lambda\mathbb{Z}^{2g} = \underset{\sim}{P}(M)\,. \end{cases}$$

Proof. Let $\sigma = (s_i) \in H^1(M,\mathbb{C}) = \mathbb{C}^{2g}$ be a cohomology class as given in Lemma 23; thus

$$t_i = \int_\tau \varphi_i = \sigma \cup (\delta\varphi_i) = \sum_{k,\ell} s_k\, \chi_{k\ell}\,\omega_{\ell i}\,,$$

or in matrix form

(19) $$t = {}^t\Omega {}^t\chi\, s\ .$$

Recalling that ${}^t\chi = -\chi$, and that $(\Omega,\bar\Omega)^{-1} = \begin{pmatrix} \bar Q^{-1}\cdot\bar\Lambda \\ Q^{-1}\cdot\Lambda\end{pmatrix}$ as in Lemma 22, rewrite (19) in the form

$$t = -{}^t\Omega\,\chi(\Omega,\bar\Omega)\begin{pmatrix} \bar Q^{-1}\cdot\bar\Lambda \\ Q^{-1}\cdot\Lambda\end{pmatrix} s = (0,iP)\begin{pmatrix} \bar Q^{-1}\cdot\bar\Lambda \\ Q^{-1}\cdot\Lambda\end{pmatrix} s = iPQ^{-1}\Lambda s\ .$$

Now by construction, $\Lambda s \in \mathbb{C}^g/\Lambda\mathbb{Z}^{2g} = \underset{\sim}{P}(M)$ represents the complex line bundle ξ ; that is to say,

$$\xi = \Lambda s = -\,iQP^{-1}t \in \mathbb{C}^g/\Lambda\mathbb{Z}^{2g} = \underset{\sim}{P}(M)\ ,$$

which is the second assertion of equation (18). The first assertion of equation (18) then follows immediately from equation (13), and the proof is thereby concluded.

There are a number of almost immediate consequences of Abel's theorem, as stated here.

Corollary 1. For any fixed point $q \in M$, the mapping $M \longrightarrow \underset{\sim}{P}(M)$ defined by $p \longrightarrow \zeta_p\zeta_q^{-1}$ is a complex analytic mapping.

Proof. It is more convenient to consider the mapping $M \longrightarrow \underset{\sim}{J}(M)$ determined by the canonical isomorphism $\underset{\sim}{J}(M) \cong \underset{\sim}{P}(M)$. By Theorem 18 this map has the form

$$p \in M \longrightarrow \left(\int_q^p \varphi_i\right) \in \mathbb{C}^g/{}^t\Omega\mathbb{Z}^{2g} = \underset{\sim}{J}(M)\ .$$

The value of the integral is obviously a complex analytic function of the upper end-point p , and hence the corollary follows at once.

Corollary 2. For any points $p_1,\dots,p_r,q_1,\dots,q_r \in M$, choose differentiable arcs τ_j from q_j to p_j on the surface M .

Then $\zeta_{p_1} \cdots \zeta_{p_r} = \zeta_{q_1} \cdots \zeta_{q_r}$ if and only if the vector $t \in \mathbb{C}^g$ with coordinates

$$t_i = \sum_{j=1}^{r} \int_{\tau_j} \varphi_i$$

belongs to the lattice ${}^t\Omega\, \mathbb{Z}^{2g}$.

Proof. By Theorem 18 the line bundle $\xi_j = \zeta_{p_j}\zeta_{q_j}^{-1}$ as an element of the Jacobi variety in given explicitly by the vector $t_j \in \mathbb{C}^g/{}^t\Omega\mathbb{Z}^{2g} = \underset{\sim}{J}(M)$ having coordinates

$$t_{ij} = \int_{\tau_j} \varphi_i \ ;$$

hence the bundle $\xi = \xi_1 \cdots \xi_r$ is given explicitly by the vector $t_i = \sum_{j=1}^{r} t_{ij}$. Now $\zeta_{p_1} \cdots \zeta_{p_r} = \zeta_{q_1} \cdots \zeta_{q_r}$ if and only if $\xi = 1$ which corresponds to the condition that $t = 0$ in $\mathbb{C}^g/{}^t\Omega\mathbb{Z}^{2g}$, that is, that $t \in {}^t\Omega\mathbb{Z}^{2g}$, as desired.

<u>Corollary 3.</u> For any points $p_1,\ldots,p_r,q_1,\ldots,q_r \in M$, choose differentiable arcs τ_j from q_j to p_j on the surface M . Then $\zeta_{p_1} \cdots \zeta_{p_r} = \zeta_{q_1} \cdots \zeta_{q_r}$ if and only if there is a closed differentiable loop τ on the surface M such that

$$\sum_{j=1}^{r} \int_{\tau_j} \varphi = \int_{\tau} \varphi$$

for all Abelian differentials $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$.

Proof. Let $\lambda_1,\ldots,\lambda_{2g}$ be closed differentiable loops on the surface M which generate the homology group $H_1(M,\mathbb{Z})$; and choose the dual basis for the cohomology group $H^1(M,\mathbb{Z})$. Then for the Abelian differentials $\varphi_i \in \Gamma(M, \mathcal{O}^{1,0})$, it follows that

$$\omega_{ki} = \int_{\lambda_k} \varphi_i\,, \quad \text{for} \quad i = 1,\dots,g \quad \text{and} \quad k=1,\dots,2g\,.$$

By Corollary 2, $\zeta_{p_1} \cdots \zeta_{p_r} = \zeta_{q_1} \cdots \zeta_{q_r}$ if and only if there is an integral vector $n = (n_k) \in \mathbb{Z}^{2g}$ such that

$$\sum_{j=1}^{r} \int_{\tau_j} \varphi_i = \sum_{k=1}^{2g} n_k \omega_{ki} = \int_{\tau} \varphi_i\,,$$

where $\tau = \sum_{k=1}^{2g} n_k \lambda_k$. Since φ_i form a basis for the Abelian differentials, the latter condition is equivalent to the condition that

$$\sum_{j=1}^{r} \int_{\tau_j} \varphi = \int_{\tau} \varphi$$

for all Abelian differentials φ, as desired.

Recalling that $\zeta_{p_1} \cdots \zeta_{p_r} \zeta_{q_1}^{-1} \cdots \zeta_{q_r}^{-1} = 1$ if and only if there is a meromorphic function f on the surface M such that

$$\mathcal{J}(f) = \sum_{i=1}^{r} 1\cdot p_i - \sum_{i=1}^{r} 1\cdot q_i\,,$$

Corollaries 2 and 3 can be restated as necessary and sufficient conditions for the existence of such meromorphic functions; this is the traditional form in which Abel's Theorem is stated.

§ 9.

(a) There are interesting structures which are finer than complex analytic structures on manifolds, but which play an important role in complex analysis. Recall from the discussion in §1(b) that the important property of complex analytic functions, for defining a complex analytic structure, is the pseudogroup property: the composition of two complex analytic local homeomorphisms is again a complex analytic local homeomorphism whenever the composition is defined. If a subset of the set of complex analytic local homeomorphisms has the pseudogroup property, then there is a further structure on complex manifolds associated to that subset. Perhaps the most interesting such subsets are defined by differential equations; these are the only such subsets which will be considered here.

Suppose that $f\colon U \longrightarrow V$ is a complex analytic local homeomorphism between two open subdomains U, V of the complex line $\mathbb{C}$; the condition that f be a local homeomorphism is of course just that $f'(z) \neq 0$ for all points $z \in U$. Introduce the differential operators θ_1, θ_2, defined as follows:

(1) $$\theta_1 f(z) = \frac{f''(z)}{f'(z)} \; ;$$

(2) $$\theta_2 f(z) = \frac{2f'(z)f'''(z) - 3f''(z)^2}{2f'(z)^2} \; .$$

Since f' is nowhere vanishing in U, the functions $\theta_\nu f$ are holomorphic throughout U. The differential operators θ_ν are of particular importance for their behaviour under the composition of mappings.

Lemma 24. Let $f: U \longrightarrow V$ and $g: V \longrightarrow W$ be complex analytic local homeomorphisms between subdomains of the complex line $\mathbf{C}$, and let $h = g \circ f$ be the composition of the two. Writing $w = f(z)$,

$$(3) \qquad \theta_\nu h(z) = \theta_\nu g(w)\cdot f'(z)^\nu + \theta_\nu f(z) \qquad \text{for } \nu = 1,2 .$$

Proof. Since $h(z) = g(w)$ and $\frac{dw}{dz} = f'(z)$, it follows from the chain rule for differentiation that

$$h'(z) = g'(w)f'(z) ,$$
$$h''(z) = g''(w)f'(z)^2 + g'(w)f''(z) ,$$
$$h'''(z) = g'''(w)f'(z)^3 + 3g''(w)f'(z)f''(z) + g'(w)f'''(z) .$$

Thus

$$\theta_1 h(z) = \frac{g''(w)f'(z)^2 + g'(w)f''(z)}{g'(w)f'(z)} = \theta_1 g(w)\cdot f'(z) + \theta_1 f(z) ;$$

and

$$\theta_2 h(z) = \frac{2g'f'[g'''(f')^3+3g''f'f''+g'f'''] - 3[(g'')^2(f')^4+2g'g''(f')^2f''+(g')^2(f'')^2]}{2(g')^2(f')^2}$$
$$= \theta_2 g(w)\cdot f'(z)^2 + \theta_2 f(z) ,$$

which completes the proof.

Now let $\mathfrak{F}_\nu$ be the family of all complex analytic local homeomorphisms f such that $\theta_\nu f(z) = 0$ at all points z where f is defined. It is an obvious consequence of Lemma 24 that the family $\mathfrak{F}_\nu$ has the pseudogroup property; this introduces the new structures next to be investigated. (It naturally occurs to one to ask why these two differential operators are selected; the reason is that they are essentially the only such operators. More precisely, one dimension, defined as the set of solutions of a system of differential equations involving only the first and higher derivatives, and possessing the pseudogroup property, then either $\mathfrak{F} = \mathfrak{F}_1$ or $\mathfrak{F} = \mathfrak{F}_2$. This is not really difficult to show, but is too much of a digression to enter into here; it is of course part of the general problem of classifying pseudogroups. For further discussion, see for instance Élie Cartan, Sur la structure des groupes infinis de transformations, Ann. Ec. Normale, 21(1904), 153-206; or the paper of Guillemin and Sternberg referred to earlier (page 4).)

The families $\mathfrak{F}_\nu$ are in fact very familiar, in a more explicit form. Firstly, if $f \in \mathfrak{F}_1$, then $f''(z) \equiv 0$, so that necessarily $f(z) = az + b$ for some constants a,b; the condition that $f'(z) \neq 0$ is merely that $a \neq 0$. Thus, $\mathfrak{F}_1$ consists of the complex affine mappings. Secondly, it is an easy calculation to show that

$$(4) \qquad \theta_2 f(z) = -2 f'(z)^{\frac{1}{2}} \frac{d^2}{dz^2} f'(z)^{-\frac{1}{2}} .$$

Therefore, if $f \in \mathfrak{F}_2$ it follows that $\frac{d^2}{dz^2} f'(z)^{-\frac{1}{2}} \equiv 0$, so that $f'(z) = (cz + d)^{-2}$ for some constants c,d; and integrating again, necessarily

$$f(z) = \frac{az + b}{cz + d}$$

for some constants a,b,c,d. The condition that $f'(z) \neq 0$ amounts merely to the condition that $ad - bc \neq 0$. Thus, $\mathfrak{F}_2$ consists of the complex projective transformations. (These are also called the linear fractional or Möbius transformations by some writers.) The differential operator θ_2 is also called the Schwarzian derivative.

Turning next to the associated structures on manifolds, let M be an arbitrary two-dimensional topological manifold, and let $\{U_\alpha, z_\alpha\}$ be a coordinate covering of M, with the coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$ in the intersections $U_\alpha \cap U_\beta$. The coordinate covering $\{U_\alpha, z_\alpha\}$ will be called an $\mathfrak{F}_\nu$ coordinate covering if all the coordinate transition functions belong to the family $\mathfrak{F}_\nu$. Two $\mathfrak{F}_\nu$ coordinate coverings will be called equivalent if their union is also an $\mathfrak{F}_\nu$ coordinate covering. It is a consequence of the pseudogroup property of $\mathfrak{F}_\nu$ that this is actually an equivalence relation; recall the discussion of complex analytic structures on page 3. An equivalence class of $\mathfrak{F}_\nu$ coordinate coverings will be called an $\mathfrak{F}_\nu$ structure on the manifold. The adjectives affine and projective will frequently be used in place of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively, in view of the explicit form of these families of mappings; thus an affine structure is an $\mathfrak{F}_1$ structure, and a projective structure is an $\mathfrak{F}_2$ structure. Note that an affine coordinate covering is also a projective coordinate covering, since $\mathfrak{F}_1 \subset \mathfrak{F}_2$; and that two equivalent affine coordinate coverings are also equivalent when considered as projective coordinate coverings, for the same reason. Hence an affine structure belongs to a well defined projective structure; the affine structure is said to be subordinate to the projective structure. In the same manner, a projective structure is subordinate to a well defined complex analytic structure. For this on a Riemann surface, meaning projective or affine structures subordinate to the given complex structure. Observe that it remains to be seen whether a complex structure actually has a subordinate projective or affine structure, and whether that subordinate structure is unique.

To investigate these questions, consider any complex analytic coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ of the Riemann surface M. Recall from §5(c) that the canonical bundle $\kappa \in H^1(M, \mathcal{O}^*)$ is defined by the cocycle $(\kappa_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$, where $\kappa_{\alpha\beta}(p) = f'_{\alpha\beta}(z_\beta(p))^{-1}$ for points $p \in U_\alpha \cap U_\beta$, and $f_{\alpha\beta}$ are the coordinate transition functions. To each intersection $U_\alpha \cap U_\beta$ associate the complex analytic function $\theta_\nu f_{\alpha\beta}(z_\beta)$ defined in $z_\beta(U_\alpha \cap U_\beta)$; and consider the element $\sigma_{\nu\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}(\kappa^\nu))$ naturally associated to that function. Thus in terms of the local coordinate mapping z_β defined in U_β,

$$\sigma_{\nu\alpha\beta}(z_\beta(p)) = \theta_\nu f_{\alpha\beta}(z_\beta(p)) ; \tag{5}$$

and if $p \in U_\gamma$ as well, then in terms of the local coordinate mapping z_γ defined in U_γ,

$$\sigma_{\nu\alpha\beta}(z_\gamma(p)) = \kappa_{\gamma\beta}(p)^\nu \cdot \sigma_{\nu\alpha\beta}(z_\beta(p)) .$$

The coordinate transition functions satisfy the condition that $f_{\alpha\gamma}(z_\gamma) = f_{\alpha\beta} \circ f_{\beta\gamma}(z_\gamma)$ for $z_\gamma \in z_\gamma(U_\alpha \cap U_\beta \cap U_\gamma)$; so by Lemma 24,

$$\theta_\nu f_{\alpha\gamma}(z_\gamma) = \theta_\nu f_{\alpha\beta}(z_\beta) \cdot f'_{\beta\gamma}(z_\gamma)^\nu + \theta_\nu f_{\beta\gamma}(z_\gamma) .$$

Rewriting the latter equation

(6) $$\sigma_{\nu\alpha\gamma}(z_\gamma(p)) = \sigma_{\nu\alpha\beta}(z_\beta(p))\cdot\kappa_{\beta\gamma}(p)^{-\nu} + \sigma_{\nu\beta\gamma}(z_\gamma(p)) \; ;$$

or equivalently,

$$\sigma_{\nu\alpha\gamma}(z_\gamma(p)) = \sigma_{\nu\alpha\beta}(z_\gamma(p)) + \sigma_{\nu\beta\gamma}(z_\gamma(p)) \; .$$

However, this means that the elements $(\sigma_{\nu\alpha\beta})$ define a cocycle $(\sigma_{\nu\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}(\kappa^\nu))$. Thus to any complex analytic coordinate covering $\mathfrak{U}$ of M there is canonically associated a cocycle

$$(\sigma_{\nu\alpha\beta}) = (\theta_\nu f_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}(\kappa^\nu)) \; , \qquad \nu = 1,2.$$

An $\mathfrak{S}_\nu$ <u>connection for the covering</u> $\mathfrak{U}$ is a zero cochain $h = (h_\alpha) \in C^0(\mathfrak{U}, \mathcal{O}(\kappa^\nu))$ such that $\delta h = \sigma_\nu$. The connections h, h' for two coverings $\mathfrak{U}, \mathfrak{U}'$ will be called <u>equivalent</u> if together they form part of a connection for the union of the two coverings. (Note that the cocycle σ_ν associated to the union of the two coverings consists of the original cocycles for the two separate coverings, in view of the fact that it is canonically defined. The equivalence condition is just that the two connections h, h' can be extended to form a connection for the union of the two coverings.) An equivalence class of connections will be called an $\mathfrak{S}_\nu$ <u>connection for the manifold</u> M. As before, an $\mathfrak{S}_1$ connection will also be called an <u>affine connection</u>, and an $\mathfrak{S}_2$ connection will also be called a <u>projective connection</u>. Explicitly, an $\mathfrak{S}_\nu$ connection for $\mathfrak{U}$ consists of sections $(h_\alpha) \in \Gamma(U_\alpha, \mathcal{O}(\kappa^\nu))$ such that

$$\sigma_{\nu\alpha\beta}(p) = h_\beta(p) - h_\alpha(p) \quad \text{for} \quad p \in U_\alpha \cap U_\beta \; .$$

in terms of the local mapping function z_α in the coordinate neighborhood U_α, the section h_α is realized as a complex analytic function $h_\alpha(z_\alpha)$ in $z_\alpha(U_\alpha)$; and the coboundary condition can be restated as

$$(7) \qquad \sigma_{\nu\alpha\beta}(z_\beta(p)) = h_\beta(z_\beta(p)) - \kappa_{\alpha\beta}(p)^{-\nu} h_\alpha(z_\alpha(p)) \quad \text{for} \quad p \in U_\alpha \cap U_\beta .$$

Theorem 19. There is a canonical one-to-one correspondence between the $\mathfrak{F}_\nu$ connections on a Riemann surface and the $\mathfrak{F}_\nu$ structures on that surface.

Proof. Let h be an $\mathfrak{F}_\nu$ connection on a Riemann surface M, and choose a representative connection $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{O}(\kappa^\nu))$ for some complex analytic coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ of M. Note first that, after passing to a refinement of the covering if necessary, there will be complex analytic homeomorphisms w_α on the sets $z_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}$ such that $h_\alpha(z_\alpha) = \theta_\nu w_\alpha(z_\alpha)$. To see this, it is only necessary to show that in some open neighborhood of any point there will exist a solution w_α of the differential equation $\theta_\nu w_\alpha = h_\alpha$, such that $w'_\alpha \neq 0$. For the case $\nu = 1$ the differential equation is the linear equation $w''_\alpha - h_\alpha w'_\alpha = 0$, which has solutions with arbitrarily prescribed values for w'_α at any point. For the case $\nu = 2$, recalling formula (4), the differential equation can be rewritten $v''_\alpha + 2h_\alpha v_\alpha = 0$ where $w'_\alpha = v_\alpha^{-2}$, and the same result holds. Note further that the most general such homeomorphism is of the form $\tilde{w}_\alpha = v_\alpha \circ w_\alpha$ for some element $v_\alpha \in \mathfrak{F}_\nu$. For if $\tilde{w}_\alpha$ is any analytic homeomorphism satisfying $h_\alpha(z_\alpha) = \theta_\nu \tilde{w}_\alpha(z_\alpha)$, then putting $v_\alpha = \tilde{w}_\alpha \circ w_\alpha^{-1}$, write $\tilde{w}_\alpha = v_\alpha \circ w_\alpha$; but by Lemma 24,

$h_\alpha = \theta_\nu \tilde{w}_\alpha = \theta_\nu(v_\alpha \circ w_\alpha) = (\theta_\nu v_\alpha)\cdot(w'_\alpha)^\nu + \theta_\nu w_\alpha = (\theta_\nu v_\alpha)\cdot(w'_\alpha)^\nu + h_\alpha$, hence $\theta_\nu v_\alpha = 0$ and $v_\alpha \in \mathcal{F}_\nu$.

Now for the given open covering $\{U_\alpha\}$, the most general complex analytic coordinate covering is of the form $\{U ,w_\alpha \circ z_\alpha\}$ for some complex analytic homeomorphisms $w_\alpha: V_\alpha \longrightarrow W_\alpha \subset \mathbb{C}$; the associated coordinate transition functions are $\tilde{f}_{\alpha\beta} = (w_\alpha \circ z_\alpha)\circ(w_\beta \circ z_\beta)^{-1} = w_\alpha \circ f_{\alpha\beta} \circ w_\beta^{-1}$, where $f_{\alpha\beta}$ are the coordinate transition functions for the covering $\{U_\alpha, z_\alpha\}$. Writing $w_\alpha \circ f_{\alpha\beta} = \tilde{f}_{\alpha\beta} \circ w_\beta$ and applying Lemma 24 again,

$$(\theta_\nu w_\alpha)(f'_{\alpha\beta})^\nu + (\theta_\nu f_{\alpha\beta}) = (\theta_\nu \tilde{f}_{\alpha\beta})(w'_\beta)^\nu + (\theta_\nu w_\beta) ,$$

or upon rewriting,

$$(\theta_\nu \tilde{f}_{\alpha\beta})(w'_\beta)^\nu = \sigma_{\nu\alpha\beta} + h_\alpha \kappa_{\alpha\beta}^{-\nu} - h_\beta ,$$

where $h_\alpha = \theta_\nu w_\alpha$. From equation (7) it then follows that $\theta_\nu \tilde{f}_{\alpha\beta} = 0$ precisely when h_α is a connection for the covering $\mathcal{U}$. Thus each $\mathcal{F}_\nu$ coordinate covering corresponds to an $\mathcal{F}_\nu$ connection; and from the observations in the preceding paragraph, this is a one-to-one correspondence. It is obvious that equivalences are preserved, hence the theorem follows as stated.

Corollary 1. An $\mathcal{F}_\nu$ coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ on a Riemann surface M represents the $\mathcal{F}_\nu$ structure canonically associated to the $\mathcal{F}_\nu$ connection h if and only if h is represented by the zero $\mathcal{F}_\nu$ connection $(h_\alpha) \in C^0(\mathcal{U} , \mathcal{O}(\kappa^\nu))$ for the covering $\mathcal{U}$.

Proof. Since the structure canonically associated to the connection h is described in terms of an analytic coordinate covering $\{U_\alpha, z_\alpha\}$ by a change of coordinates by homeomorphisms w_α satisfying $\theta_\nu w_\alpha = h_\alpha$, this is entirely obvious.

Note that if $h = (h_\alpha)$ is an $\mathcal{F}_\nu$ connection, then from equation (7) it follows that the most general $\mathcal{F}_\nu$ connection is $h+g$ where $g \in \Gamma(M, \mathcal{O}(\kappa^\nu))$ is an arbitrary section; thus the set of $\mathcal{F}_\nu$ connections, if non-empty, form a complex linear manifold of dimension $\gamma(\kappa^\nu)$. And applying Theorem 19, if a Riemann surface M admits any $\mathcal{F}_\nu$ structure, then the set of all structures form in a canonical manner a complex linear space of dimension $\gamma(\kappa^\nu)$. For affine structures this dimension is $\gamma(\kappa) = g$; and for projective structures this dimension is $\gamma(\kappa^2) = 3g - 3$, by the Riemann-Roch theorem.

As for the existence of an $\mathcal{F}_\nu$ connection, it is clear from the definition of a connection that the necessary and sufficient condition is that $\sigma_\nu = 0$ in $H^1(M, \mathcal{O}(\kappa^\nu))$, where σ_ν is the cohomology class defined by the cocycle $\sigma_{\nu\alpha\beta} = \theta_\nu f_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{O}(\kappa^\nu))$. Recalling the preceding investigations of these cohomology groups, the following existence theorems arise.

Corollary 2. A compact Riemann surface of genus $g > 1$ always admits projective structures.

Proof. By the Serre duality theorem $H^1(M, \mathcal{O}(\kappa^2)) \cong \Gamma(M, \mathcal{O}(\kappa^{-1}))$; but since $c(\kappa^{-1}) = -c(\kappa) = 2-2g$, it follows that $c(\kappa^{-1}) < 0$ for $g > 1$, and therefore that $H^1(M, \mathcal{O}(\kappa^2)) \cong \Gamma(M, \mathcal{O}(\kappa^{-1})) = 0$, which suffices to prove the assertion.

Corollary 3. A compact Riemann surface admits affine structures if and only if $c(\kappa) = 0$, hence if and only if the surface has genus one.

Proof. By the Serre duality theorem again, $H^1(M, \mathcal{O}(\kappa))$ is canonically dual to $\Gamma(M, \mathcal{O}) = \mathbb{C}$; hence there is an isomorphism $H^1(M, \mathcal{O}(\kappa)) \cong \mathbb{C}$. Select a coordinate covering $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ so that the cohomology class $\sigma_1 \in H^1(M, \mathcal{O}(\kappa))$ is represented by a cocycle $\sigma_{1\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}(\kappa))$; recall that

$$\sigma_{1\alpha\beta} = \theta_1 f_{\alpha\beta} = \frac{f''_{\alpha\beta}}{f'_{\alpha\beta}} = \frac{d}{dz_\beta} \log f'_{\alpha\beta} = -\frac{d}{dz_\beta} \log \kappa_{\alpha\beta} .$$

Considering $\sigma_{1\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^{1,0})$, then $\sigma_{1\alpha\beta} dz_\beta = -d(\log \kappa_{\alpha\beta})$. But this is the same form of cocycle considered in the proof of Lemma 14; so, applying the arguments there (especially on page 102), it follows readily that under the chosen explicit form of Serre's duality, the cohomology class σ_1 corresponds to the constant $-2\pi i\, c(\kappa)$. Hence $\sigma_1 = 0$ if and only if $c(\kappa) = 0$, which suffices to prove the assertion.

It was demonstrated earlier (page 115) that $\mathbb{P}^1$ is the only compact Riemann surface of genus zero, and it obviously has a projective structure; this case is essentially trivial, and will henceforth be excluded from consideration. The Riemann surfaces of genus 1 are the only compact Riemann surfaces which admit affine structures, as a consequence of Corollary 3. This can actually be seen directly as follows. If $\{U_\alpha, z_\alpha\}$ is an affine coordinate covering, the transition functions are of the form $z_\alpha = a_{\alpha\beta} z_\beta + b_{\alpha\beta}$

for some constants $a_{\alpha\beta}, b_{\alpha\beta} \in \mathbf{C}$; and so the canonical bundle is defined by the functions $\kappa_{\alpha\beta} = (dz_\alpha/dz_\beta)^{-1} = a_{\alpha\beta}^{-1}$. As in §8(a), line bundles defined by a constant cocycle necessarily have Chern class zero, hence $0 = c(\kappa) = 2g-2$ and so $g = 1$. An affine structure is itself a projective structure; and we shall later see that in this case there is a one-to-one correspondence between affine and projective structures. In general, for Riemann surfaces of genus $g > 1$, there are no affine structures at all, by Corollary 3; but each surface admits a family of projective structures, by Corollary 2.

(b) The families $\mathfrak{F}_\nu$ can be considered as groups, rather than merely as pseudogroups. In view of Corollary 3 of Theorem 19, for the remainder of the discussion here we shall consider explicitly only the family $\mathfrak{F}_2$ of projective transformations; the reader can readily provide corresponding statements for the family $\mathfrak{F}_1$ of affine transformations. Viewing a projective transformation as a complex analytic homeomorphism $\varphi\colon \mathbb{P} \longrightarrow \mathbb{P}$ (as discussed in L. Ahlfors, Complex Analysis, (McGraw-Hill, 1966), for example), compositions are well-defined for any two projective transformations, so the set of all such form a group; this group is called the projective linear group of rank 2 over the complex numbers, and will be denoted by $PL(2,\mathbf{C})$.

The projective structures on a surface can in a sense be described by a slight modification of the cohomological machinery which has been used earlier. Let $\mathfrak{U} = \{U_\alpha\}$ be an open covering of the topological surface M , and G be any abstract group,

(not necessarily commutative). A q-cochain of the covering $\mathcal{U}$ with coefficients in G is a function φ which associates to any q-simplex $(U_{\alpha_0},\ldots,U_{\alpha_q}) \in N(\mathcal{U})$ an element $\varphi_{\alpha_0\ldots\alpha_q} = \varphi(U_{\alpha_0},\ldots,U_{\alpha_q}) \in G$; the set of all such q-cochains will be denoted by $C^q(\mathcal{U},G)$, but this is now viewed merely as a set, with no specified group structure. A one-cochain $(\varphi_{\alpha\beta}) \in C^1(\mathcal{U},G)$ is called a one-cocycle if $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ and $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$; and the set of one-cocycles will be denoted by $Z^1(\mathcal{U},G)$. Two cocycles $(\varphi_{\alpha\beta}),(\psi_{\alpha\beta})$ will be called equivalent if there is a zero cochain $(\theta_\alpha) \in C^0(\mathcal{U},G)$ such that $\psi_{\alpha\beta} = \theta_\alpha\varphi_{\alpha\beta}\theta_\beta^{-1}$; the set of equivalence classes will be denoted by $H^1(\mathcal{U},G)$, and will be called the one-dimensional cohomology set of $\mathcal{U}$ with coefficients in G . If $\mathcal{V}$ is a refinement of the covering $\mathcal{U}$, with refining mapping $\mu\colon \mathcal{V} \longrightarrow \mathcal{U}$, then as in §3(b) there are induced mappings $\mu\colon C^q(\mathcal{U},G) \longrightarrow C^q(\mathcal{V},G)$. It is easy to see that these lead to a mapping $\mu^*\colon H^1(\mathcal{U},G) \longrightarrow H^1(\mathcal{V},G)$; verification will be left to the reader. Further, as an analogue of Lemma 5, if $\mu\colon \mathcal{V} \longrightarrow \mathcal{U}$ and $\nu\colon \mathcal{V} \longrightarrow \mathcal{U}$ are two refining mappings, then $\mu^* = \nu^*$. (For if $(\varphi_{\alpha\beta}) = (\varphi(U_\alpha,U_\beta)) \in Z^1(\mathcal{U},G)$, then define a zero-cochain $(\theta_\alpha) \in C^0(\mathcal{V},G)$ by $\theta_\alpha = \theta(V_\alpha) = \varphi(\mu V_\alpha,\nu V_\alpha)$. Then $\mu^*\varphi(V_\alpha,V_\beta) = \varphi(\mu V_\alpha,\mu V_\beta) =$
$= \varphi(\mu V_\alpha,\nu V_\alpha)\varphi(\nu V_\alpha,\nu V_\beta)\varphi(\nu V_\beta,\mu V_\beta) = \theta_\alpha\cdot\nu^*\varphi(V_\alpha,V_\beta)\cdot\theta_\beta^{-1}$, so that $\mu^*\varphi$ is equivalent to $\nu^*\varphi$.) Then put

$$H^1(M,G) = \mathrm{dir.lim.}_{\mathcal{U}}\, H^1(\mathcal{U},G) ,$$

to define the first cohomology set of M with coefficients in the group G .

Lemma 25. There is a canonical mapping from the set of projective structures on a surface M into the cohomology set $H^1(M, PL(2,\mathbb{C}))$.

Proof. For any projective structure select a representative projective coordinate covering $\{U_\alpha, z_\alpha\}$, with coordinate transition functions $(\varphi_{\alpha\beta})$. The elements $\varphi_{\alpha\beta}$ can be considered in two ways: either as mappings $\varphi_{\alpha\beta}$: $z_\beta(U_\alpha \cap U_\beta) \longrightarrow z_\alpha(U_\alpha \cap U_\beta)$, or as elements $\varphi_{\alpha\beta} \in PL(2,\mathbb{C})$ associated to non-empty intersections $U_\alpha \cap U_\beta \subset M$. In either case $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$, and $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, so that $(\varphi_{\alpha\beta})$ determine a cocycle in $Z^1(\mathfrak{U}, PL(2,\mathbb{C}))$. If $\{U_\alpha, \tilde{z}_\alpha\}$ is an equivalent projective coordinate covering defined in terms of the same open covering $\mathfrak{U}$ of M and having coordinate transition functions $(\tilde{\varphi}_{\alpha\beta})$, then there are elements $\theta_\alpha \in PL(2,\mathbb{C})$ such that $\tilde{\varphi}_{\alpha\beta} = \theta_\alpha \varphi_{\alpha\beta} \theta_\beta^{-1}$; so the cocycles $(\varphi_{\alpha\beta})$ and $(\tilde{\varphi}_{\alpha\beta})$ are equivalent. There is thus a well-defined mapping, from equivalence classes of projective coordinate coverings of M defined in terms of the open covering $\mathfrak{U}$, into the cohomology set $H^1(\mathfrak{U}, PL(2,\mathbb{C}))$. A projective coordinate covering $\{U_\alpha, z_\alpha\}$ induces a natural projective coordinate covering for any refinement $\mathfrak{V} < \mathfrak{U}$, and this is evidently compatible with the cohomology mappings $H^1(\mathfrak{U}, PL(2,\mathbb{C})) \longrightarrow H^1(\mathfrak{V}, PL(2,\mathbb{C}))$. Finally, two projective coordinate coverings $\{U_\alpha, z_\alpha\}$ and $\{\tilde{U}_\alpha, \tilde{z}_\alpha\}$ are equivalent if and only if they induce equivalent projective coordinate coverings for a common refinement $\mathfrak{V}$ of $\mathfrak{U}$ and $\tilde{\mathfrak{U}}$. This serves to conclude the proof.

The element of $H^1(M, PL(2,\mathbb{C}))$ corresponding to a projective structure will be called the <u>coordinate (cohomology) class</u> of that structure. The mapping which associates to a projective structure its coordinate cohomology class $\in H^1(M, PL(2,\mathbf{C}))$ is neither one-to-one nor onto. However, restricting consideration to the projective structures subordinate to a fixed complex structure, the mapping is one-to-one, in the sense that two projective structures on a Riemann surface are equivalent when they have the same coordinate cohomology class. Before turning to the proof of this assertion, it is convenient to introduce some further terminology.

Again consider an abstract group G, but now suppose that G acts as a group of homeomorphisms on a topological space S. For any cohomology class $\varphi \in H^1(M,G)$, select a basis $\mathcal{U} = \{U_\alpha\}$ for the open sets of the topological surface M and a representative cocycle $(\varphi_{\alpha\beta}) \in Z^1(\mathcal{U}, G)$ for the cohomology class φ. To each set $U_\alpha \in \mathcal{U}$ associate the set $\mathcal{S}_\alpha$ of continuous mappings from U_α into S; and to each inclusion $U_\alpha \subset U_\beta$ associate the function $\rho_{\beta\alpha}: \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\beta$ which takes a mapping $z_\alpha \in \mathcal{S}_\alpha$ into the mapping $\rho_{\beta\alpha} z_\alpha \in \mathcal{S}_\beta$ defined by

$$(\rho_{\beta\alpha} z_\alpha)(p) = \varphi_{\beta\alpha}(z_\alpha(p)) \qquad \text{for } p \in U_\alpha \subset U_\beta .$$

Since $\varphi_{\beta\alpha}: S \longrightarrow S$ is continuous, this definition makes sense. Whenever $U_\alpha \subset U_\beta \subset U_\gamma$ and $z_\alpha \in \mathcal{S}_\alpha$, it follows readily from the cocycle condition on $(\varphi_{\alpha\beta})$ that $\rho_{\gamma\beta}\rho_{\beta\alpha} = \rho_{\gamma\alpha}$. Therefore $\{\mathcal{U}, \mathcal{S}_\alpha, \rho_{\alpha\beta}\}$ is a presheaf of sets over M, which is easily seen to be complete; the associated sheaf will be called the <u>sheaf of germs of continuous sections of</u> φ <u>with values in</u> S, and will be denoted by $\mathcal{C}(\varphi, S)$. A section $z = (z_\alpha) \in \Gamma(M, \mathcal{C}(\varphi,S))$ corresponds to a family of continuous mappings $z_\alpha: U_\alpha \longrightarrow S$ such that

$z_\alpha(p) = \varphi_{\alpha\beta}(z_\beta(p))$ whenever $p \in U_\alpha \cap U_\beta$. If S has a complex structure, and G acts as a group of complex analytic homeomorphisms of S, then in the same manner we can define the sheaf of germs of complex analytic sections of φ with values in S, a sheaf which will be denoted by $\mathcal{O}(\varphi,S)$. If S has an algebraic structure, and G acts as a group of automorphisms of that structure, then the sheaves $\mathcal{C}(\varphi,S)$ and $\mathcal{O}(\varphi,S)$ can be given the structures of sheaves of those algebraic structures. The space S will be dropped from the notation if there is no danger of confusion. Two examples will be of particular interest here. The first is that in which $G = GL(n,\mathbb{C})$ and $S = \mathbb{C}^n$; G acts as a group of complex analytic isomorphisms of the complex vector space $\mathbb{C}^n$. A cohomology class $\varphi \in H^1(M, GL(n,\mathbb{C}))$ will be called a flat complex vector bundle of rank n over M; the corresponding sheaf $\mathcal{O}(\varphi,\mathbb{C}^n)$ of complex analytic sections has the structure of a sheaf of complex vector spaces. The case $n = 1$ is just the case of flat complex line bundles, as considered earlier. Of course, the group $SL(n,\mathbb{C})$ can be used in place of $GL(n,\mathbb{C})$. The second is that in which $G = PL(2,\mathbb{C})$ and $S = \mathbb{P}$. A cohomology class $\varphi \in H^1(M,PL(2,\mathbb{C}))$ will be called a flat projective line bundle over M.

Note, by the way, that if $\varphi \in H^1(M,PL(2,\mathbb{C}))$ is the coordinate cohomology class of a projective structure on M, and if $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ is a projective coordinate covering with coordinate transition functions $(\varphi_{\alpha\beta}) \in Z^1(\mathfrak{U}, PL(2,\mathbb{C}))$ which represent the cohomology class φ, then actually $(z_\alpha) \in \Gamma(M, \mathcal{C}(\varphi,\mathbb{P}))$. Indeed, the sections z_α have the further property that they are local homeomorphisms from M into $\mathbb{P}$. Conversely if $\varphi \in H^1(M,PL(2,\mathbb{C}))$

has sections $(z_\alpha) \in \Gamma(M, \mathcal{C}(\varphi, \mathbb{P}))$ which are local homeomorphisms, then those sections define a projective coordinate covering of M upon suitable refinement, and φ is the coordinate cohomology class of that projective structure. The subset of $\Gamma(M, \mathcal{C}(\varphi, \mathbb{P}))$ consisting of sections which are local homeomorphisms will be called the set of coordinate sections, and will be denoted by $\Gamma_0(M, \mathcal{C}(\varphi, \mathbb{P}))$. Thus a cohomology class $\varphi \in H^1(M, PL(2, \mathbb{C}))$ is the coordinate class of a projective structure on the topological surface M if and only if there exists a coordinate section $(z_\alpha) \in \Gamma_0(M, \mathcal{C}(\varphi, \mathbb{P}))$; the set of all coordinate sections, modulo the obvious equivalence relation, correspond to all projective structures on M with the given coordinate cohomology class. And similarly, φ is the coordinate class of a projective structure on the Riemann surface M if and only if there exists a coordinate section $(z_\alpha) \in \Gamma_0(M, \mathcal{O}(\varphi, \mathbb{P}))$; the set of all analytic coordinate sections, modulo the obvious equivalence, correspond to all projective structures subordinate to the given complex analytic structure on M and with the given coordinate cohomology class.

To any matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ there corresponds a projective transformation $\varphi = \varphi_T \in PL(2, \mathbb{C})$, of the form

(8) $$\varphi(z) = \frac{az + b}{cz + d} ;$$

every transformation $\varphi \in PL(2, \mathbb{C})$ can be so represented, and two matrices T, T' represent the same projective transformation if and only if $T' = \pm T$. The mapping $\mu: T \longrightarrow \varphi_T$ is a group homomorphism, forming part of the exact sequence of groups

$$(9) \qquad 0 \longrightarrow (\pm I) \longrightarrow SL(2,\mathbb{C}) \xrightarrow{\mu} PL(2,\mathbb{C}) \longrightarrow 0 ,$$

where 0 stands for the trivial group. (Since $PL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/(\pm I)$, this provides the projective linear group with the structure of a complex Lie group.) For any covering $\mathcal{U} = \{U_\alpha\}$ of M, the homomorphism μ clearly induces a mapping $\mu: C^1(\mathcal{U}, SL(2,\mathbb{C})) \longrightarrow C^1(\mathcal{U}, PL(2,\mathbb{C}))$; it is easy to see that this mapping takes cocycles into cocycles and preserves equivalence classes, hence induces a mapping

$$(10) \qquad \mu^*: H^1(M,SL(2,\mathbb{C})) \longrightarrow H^1(M,PL(2,\mathbb{C})) .$$

<u>Lemma 26.</u> If M is a compact Riemann surface and $\varphi \in H^1(M,PL(2,\mathbb{C}))$ is the coordinate cohomology class of a projective structure on M, then there exists a cohomology class $T \in H^1(M,SL(2,\mathbb{C}))$ such that $\varphi = \mu^*(T)$. Further, if $(z_\alpha) \in \Gamma_0(M, \mathcal{O}(\varphi,\mathbb{P}))$ is any coordinate section, then there exist a flat complex line bundle $\xi \in H^1(M,\mathbb{C}^*)$ and a section $h = (h_{1\alpha},h_{2\alpha}) \in \Gamma(M, \mathcal{O}(\xi T,\mathbb{C}^2))$ such that $z_\alpha = h_{1\alpha}/h_{2\alpha}$.

<u>Remark.</u> It is clear that whenever $\xi \in H^1(M,\mathbb{C}^*)$ and $T \in H^1(M,SL(2,\mathbb{C}))$, the product $\xi T \in H^1(M,GL(2,\mathbb{C}))$ is well defined.

Proof. If $\varphi \in H^1(M,PL(2,\mathbb{C}))$ is the coordinate cohomology class of a projective structure on M, select a projective coordinate covering $\mathcal{U} = \{U_\alpha, z_\alpha\}$ with coordinate transition functions $(\varphi_{\alpha\beta}) \in Z^1(\mathcal{U}, PL(2,\mathbb{C}))$ which represent the cohomology class φ. For each element $\varphi_{\alpha\beta}$ select a matrix $T_{\alpha\beta} \in SL(2,\mathbb{C})$ such that $\varphi_{\alpha\beta} = \mu(T_{\alpha\beta})$; so the coordinate transitions can be written

$$(11) \qquad z_\alpha(p) = \frac{a_{\alpha\beta} z_\beta(p) + b_{\alpha\beta}}{c_{\alpha\beta} z_\beta(p) + d_{\alpha\beta}} \qquad \text{for} \quad p \in U_\alpha \cap U_\beta ,$$

where

$$T_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} .$$

Note that the canonical bundle κ is defined by

$$\kappa_{\alpha\beta}(p) = \left(\frac{dz_\alpha}{dz_\beta}\right)^{-1} = (c_{\alpha\beta} z_\beta(p) + d_{\alpha\beta})^2 \quad \text{for} \quad p \in U_\alpha \cap U_\beta .$$

Select any $g-1$ distinct points $p_1, p_2, \ldots, p_{g-1} \in M$, where g is the genus of M; and introduce the complex line bundle $\zeta = \Pi_i \zeta_{p_i}^2$, where ζ_{p_i} are the point bundles of §7(c). Write $\zeta = \eta\kappa$ for some complex line bundle η; since $c(\zeta) = 2(g-1) = c(\kappa)$, then $c(\eta) = 0$ so we can suppose that $\eta \in H^1(M, \mathbf{C}^*)$. Now there is an analytic section $g = (g_\alpha) \in \Gamma(M, \mathcal{O}(\zeta)) = \Gamma(M, \mathcal{O}(\eta\kappa))$ such that $\mathcal{J}(g) = \Sigma_i\, 2 \cdot p_i$; the condition that g be a section is just that

$$g_\alpha(p) = \eta_{\alpha\beta}(c_{\alpha\beta} z_\beta(p) + d_{\alpha\beta})^2 g_\beta(p) \qquad \text{for} \quad p \in U_\alpha \cap U_\beta ,$$

where $(\eta_{\alpha\beta})$ are constants representing the cohomology class η. Since the divisor of g is even, in each coordinate neighborhood U_α we can select a well-defined branch of $h_{2\alpha}(p) = \sqrt{g_\alpha(p)}$. The functions $h_{2\alpha}$ are analytic in U_α; and

$$(12) \qquad h_{2\alpha}(p) = \xi_{\alpha\beta}(c_{\alpha\beta} z_\beta(p) + d_{\alpha\beta}) h_{2\beta}(p) \qquad \text{for} \quad p \in U_\alpha \cap U_\beta ,$$

for some constants $\xi_{\alpha\beta}$. Introduce further the analytic functions $h_{1\alpha}(p) = z_\alpha(p) h_{2\alpha}(p)$. It follows readily from (11), (12), and these definitions, that

$$(13) \quad \begin{cases} h_{1\alpha}(p) = \xi_{\alpha\beta}(a_{\alpha\beta}h_{1\beta}(p) + b_{\alpha\beta}h_{2\beta}(p)) \\ h_{2\alpha}(p) = \xi_{\alpha\beta}(c_{\alpha\beta}h_{1\beta}(p) + d_{\alpha\beta}h_{2\beta}(p)) \end{cases} \quad \text{for } p \in U_\alpha \cap U_\beta .$$

The analytic functions $h_{1\alpha}, h_{2\alpha}$ are clearly linearly independent. So it follows immediately from (13) that the matrices $S_{\alpha\beta} = \xi_{\alpha\beta}T_{\alpha\beta}$ satisfy the cocycle condition, or in other words, represent an element $S \in H^1(M,GL(2,\mathbb{C}))$; and it is evident that $\mu^*(s) = \varphi$. Dividing all the matrices by their determinants will then yield a cohomology class in $H^1(M,SL(2,\mathbb{C}))$ satisfying the desired conditions. Then, for the second part of the lemma, write the coordinate transition functions for the coordinate section (z_α) in the form (11), where $T = (T_{\alpha\beta}) \in H^1(M,SL(2,\mathbb{C}))$, and repeat the above part of the proof. The constants $(\xi_{\alpha\beta})$ define a flat line bundle, since both $(T_{\alpha\beta})$ and $(\xi_{\alpha\beta}T_{\alpha\beta})$ satisfy the cocycle condition; and by (13) the functions $h_\alpha = (h_{1\alpha}, h_{2\alpha})$ determine a section in $\Gamma(M, \mathcal{O}(\xi T, \mathbb{C}^2))$, while by construction $z_\alpha(p) = h_{1\alpha}(p)/h_{2\alpha}(p)$. That concludes the proof.

Theorem 20. On a compact Riemann surface, the projective structures subordinate to the given complex structure are determined uniquely by their coordinate cohomology clsses.

Proof. Let $\{U_\alpha, z_\alpha\}$ and $\{U_\alpha, w_\alpha\}$ be two projective coordinate coverings having the same coordinate cohomology class φ ; there is no loss of generality in assuming that the two coverings actually have the same coordinate transition functions $(\varphi_{\alpha\beta}) \in Z^1(\mathfrak{U}, PL(2,\mathbb{C}))$. Applying Lemma 26, select a cohomology class $T \in H^1(M,SL(2,\mathbb{C}))$ such that $\mu^*(T) = \varphi$, and suppose that

$(T_{\alpha\beta}) \in Z^1(\mathcal{U}, SL(2,\mathbf{C}))$ is a representative cocycle. Further, select flat complex line bundles $\xi = (\xi_{\alpha\beta})$ and $\eta = (\eta_{\alpha\beta})$, together with complex analytic sections $g = (g_{1\alpha}, g_{2\alpha}) \in \Gamma(M, \mathcal{O}(\xi T, \mathbf{C}^2))$ and $h = (h_{1\alpha}, h_{2\alpha}) \in \Gamma(M, \mathcal{O}(\eta T, \mathbf{C}^2))$, such that $z_\alpha(p) = g_{1\alpha}(p)/g_{2\alpha}(p)$ and $w_\alpha(p) = h_{1\alpha}(p)/h_{2\alpha}(p)$. The matrices

$$F_\alpha(p) = \begin{pmatrix} g_{1\alpha}(p) & h_{1\alpha}(p) \\ g_{2\alpha}(p) & h_{2\alpha}(p) \end{pmatrix}$$

are complex analytic functions in each set U_α, and

$$F_\alpha(p) = T_{\alpha\beta} \cdot F_\beta(p) \cdot \begin{pmatrix} \xi_{\alpha\beta} & 0 \\ 0 & \eta_{\alpha\beta} \end{pmatrix} \quad \text{for} \quad p \in U_\alpha \cap U_\beta .$$

Therefore the functions $\det F_\alpha$ are complex analytic in each set U_α, and

$$\det F_\alpha(p) = \xi_{\alpha\beta} \cdot \eta_{\alpha\beta} \cdot \det F_\beta(p) \quad \text{for } p \in U_\alpha \cap U_\beta ;$$

that is to say, $(\det F_\alpha) \in \Gamma(M, \mathcal{O}(\xi\eta))$. Since $c(\xi\eta) = 0$, either $\det F_\alpha$ vanishes identically, or $\det F_\alpha$ is nowhere vanishing, by Theorem 11. From the proof of Lemma 26, recall that the functions $g_{1\alpha}$ and $g_{2\alpha}$ for instance both vanish at $g-1$ points; and therefore necessarily $\det F_\alpha \equiv 0$. This condition means that the vectors $(g_{1\alpha}, g_{2\alpha})$ and $(h_{1\alpha}, h_{2\alpha})$ are everywhere linearly dependent, or equivalently that $z_\alpha = w_\alpha$ everywhere. Therefore the two projective coordinate coverings coincide, and the desired result has been demonstrated.

This theorem shows that the mapping which associates to the projective structures on a Riemann surface their coordinate cohomology classes is one-to-one; hence the coordinate classes can be used to describe the set of projective structures on a given Riemann surface. This result is definitely false for projective structures on a topological surface; a construction of L. Bers ("Simultaneous uniformization," Bull. Amer. Math. Soc. 66(1960), pp. 94-97) provides coordinate classes with two different projective structures, such that the underlying complex analytic structures can be any two arbitrary Riemann surfaces of the same genus. There now remains the problem of determining the subset of $H^1(M,PL(2,\mathbb{C}))$ consisting of the coordinate cohomology classes of the possible projective structures on a Riemann surface M. Before approaching this problem, as a slight digression we shall consider another description of the cohomology sets $H^1(M,PL(2,\mathbb{C}))$.

The cohomology sets $H^1(M,G)$ for any group G can be described as follows. Again consider an open covering $\mathfrak{U} = \{U_\alpha\}$ of the space M. A chain of the covering $\mathfrak{U}$ based at $U_o \in \mathfrak{U}$ is a finite sequence $\gamma = (U_{\alpha_o}, U_{\alpha_1}, \ldots, U_{\alpha_m})$ of elements $U_{\alpha_i} \in \mathfrak{U}$, such that $U_{\alpha_o} = U_o$ and $U_{\alpha_{i-1}} \cap U_{\alpha_i} \neq \emptyset$ for $i = 1,\ldots,m$; and the chain is said to be closed if $U_{\alpha_m} = U_o$ also. A simple jerk on such a chain consists either in replacing a pair $U_{\alpha_i}, U_{\alpha_{i+1}}$ of consecutive elements of the chain by a triple $U_{\alpha_i}, U_\beta, U_{\alpha_{i+1}}$ when $U_{\alpha_i} \cap U_\beta \cap U_{\alpha_{i+1}} \neq \emptyset$, or in performing the inverse operation. Two chains are called homotopic if it is possible to pass from one to another by a finite sequence of simple jerks; this is clearly an equivalence relation, and the set of equivalence classes will be denoted by $\pi_1(\mathfrak{U})$. If $(U_{\alpha_0},\ldots,U_{\alpha_m})$ and $(\tilde{U}_{\alpha_0},\ldots,\tilde{U}_{\alpha_n})$ are closed chains, their product is defined to be the closed chain $(U_{\alpha_0},\ldots,U_{\alpha_m},\tilde{U}_{\alpha_0},\ldots,\tilde{U}_{\alpha_n})$; it is clear that this product can be carried over to the set $\pi_1(\mathfrak{U})$ of equivalence classes of closed chains, and that $\pi_1(\mathfrak{U})$ is then a group. (The associativity property is obvious; the identity is the equivalence class represented by the closed chain (U_0) , and the inverse of the equivalence class represented by a chain $(U_{\alpha_0},\ldots,U_{\alpha_m})$ is represented by the chain $(U_{\alpha_m},U_{\alpha_{m-1}},\ldots,U_{\alpha_0})$. Details will be left to the reader.) The group depends of course on the choice of the base point U_0 , and the notation $\pi_1(\mathfrak{U},U_0)$ will be used when it is necessary to specify the base point. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, with refining mapping $\mu\colon \mathfrak{V} \longrightarrow \mathfrak{U}$, and if there is a set $V_0 \in \mathfrak{V}$ such that $\mu V_0 = U_0$, then μ induces a homomorphism $\mu^*\colon \pi_1(\mathfrak{V},V_0) \longrightarrow \pi_1(\mathfrak{U},U_0)$. (For any closed chain $(V_{\alpha_0},V_{\alpha_1},\ldots,V_{\alpha_m})$ based at V_0 set $\mu(V_{\alpha_0},V_{\alpha_1},\ldots,V_{\alpha_m}) = (\mu V_{\alpha_0},\mu V_{\alpha_1},\ldots,\mu V_{\alpha_m})$; it is evident that this mapping preserves equivalence classes, and defines the desired homomorphism μ^* .) Moreover, if $\nu\colon \mathfrak{V} \longrightarrow \mathfrak{U}$ is another such mapping, then $\mu^* = \nu^*$. (To see this, it is sufficient to observe that for any closed chain $(V_{\alpha_0},\ldots,V_{\alpha_m})$ based at V_0 and any index $r = 1,2,\ldots,m-1$, it is possible to pass from the chain $(\mu V_{\alpha_0},\ldots,\mu V_{\alpha_{r-1}},\nu V_{\alpha_r},\ldots,\nu V_{\alpha_m})$ to the chain $(\mu V_{\alpha_0},\ldots,\mu V_{\alpha_r},\nu V_{\alpha_{r-1}},\ldots,\nu V_{\alpha_m})$ by the succession of simple jerks $(\mu V_{\alpha_{r-1}},\nu V_{\alpha_r}) \longrightarrow (\mu V_{\alpha_{r-1}},\mu V_{\alpha_r},\nu V_{\alpha_r})$ and

$(\mu V_{\alpha_r}, \nu V_{\alpha_r}, \nu V_{\alpha_{r+1}}) \longrightarrow (\mu V_{\alpha_r}, \nu V_{\alpha_{r+1}})$.) Now select a fixed point $p \in M$, and consider the family consisting of open coverings $\mathcal{U} = \{U_\alpha\}$ together with a fixed element $U_o \in \mathcal{U}$ such that $p \in U_o$. This is a directed set, defining $(\mathcal{V}, V_o) < (\mathcal{U}, U_o)$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$ with a refining mapping $\mu: \mathcal{V} \longrightarrow \mathcal{U}$ so that $\mu(V_o) = U_o$. Then put

$$\pi_1(M,p) = \text{inv.lim.}_{\{\mathcal{U}, U_o\}} \pi_1(\mathcal{U}, U_o) .$$

(Recall that the inverse limit group is the subgroup of the direct product $\Pi(\pi_1(\mathcal{U}, U_o))$ consisting of those elements $(\gamma_{\mathcal{U}, U_o}) \in \Pi(\pi_1(\mathcal{U}, U_o))$ such that $\mu^*(\gamma_{\mathcal{V}, V_o}) = \gamma_{\mathcal{U}, U_o}$ whenever $(\mathcal{V}, V_o) < (\mathcal{U}, U_o)$.) It should be remarked that a change of the base point has the effect of an inner automorphism on the fundamental group $\pi_1(M)$.

Now for any abstract group G, let $\text{Hom}(\pi_1(M,p),G)$ be the set of homomorphisms from $\pi_1(M,p)$ into G. To any pair of elements $\chi \in \text{Hom}(\pi_1(M,p),G)$ and $g \in G$ there is associated another group homomorphism $\chi^g \in \text{Hom}(\pi_1(M,p),G)$ defined by $\chi^g(\pi) = g^{-1}\chi(\pi)g$ for $\pi \in \pi_1(M,p)$; two elements $\chi, \tilde{\chi} \in \text{Hom}(\pi_1(M,p),G)$ are called equivalent if $\tilde{\chi} = \chi^g$ for some g; and the set of equivalence classes will be denoted by

$$\text{Hom}(\pi_1(M,p),G)/G .$$

Actually of course, the mapping $\chi \longrightarrow \chi^g$ exhibits G as a group of operators on $\text{Hom}(\pi_1(M,p),G)$, and the above set of equivalence classes is merely the quotient space under this group action.

Lemma 27. For any surface M and any group G, there is a natural one-to-one correspondence between the cohomology set $H^1(M,G)$ and the set $\text{Hom}(\pi_1(M,p),G)/G$.

Proof. Let $\mathfrak{U} = \{U_\alpha\}$ be an open covering of M, with base point U_o containing p. For any cohomology class $\varphi \in H^1(\mathfrak{U}, G)$ select a representative cocycle $(\varphi_{\alpha\beta}) \in Z^1(\mathfrak{U}, G)$; and for any chain $(U_{\alpha_o}, U_{\alpha_1}, \ldots, U_{\alpha_m})$ based at U_o define

$$(14) \qquad \varphi(U_{\alpha_o}, U_{\alpha_1}, \ldots, U_{\alpha_m}) = \varphi_{\alpha_o\alpha_1} \cdot \varphi_{\alpha_1\alpha_2} \cdot \ldots \cdot \varphi_{\alpha_{m-1}\alpha_m} .$$

Since φ is a cocycle, this last expression is unchanged under simple jerks; for if $U_{\alpha_i} \cap U_\beta \cap U_{\alpha_{i+1}} \neq \emptyset$, then $\varphi_{\alpha_i\alpha_{i+1}} = \varphi_{\alpha_i\beta} \cdot \varphi_{\beta\alpha_{i+1}}$. Thus (14) defines a mapping

$$\chi_g \colon \pi_1(\mathfrak{U}, U_o) \longrightarrow G ,$$

which is readily seen to be a group homomorphism. If $(\varphi_{\alpha\beta})$ and $(\psi_{\alpha\beta})$ are equivalent cocycles, then there are elements $\theta_\alpha \in G$ such that $\psi_{\alpha\beta} = \theta_\alpha \varphi_{\alpha\beta} \theta_\beta^{-1}$; so for any closed chain,

$$(15) \qquad \psi(U_{\alpha_o}, U_{\alpha_1}, \ldots, U_{\alpha_m}) = \theta_{\alpha_o} \cdot \varphi(U_{\alpha_o}, U_{\alpha_1}, \ldots, U_{\alpha_m}) \cdot \theta_{\alpha_o}^{-1} ,$$

since $U_{\alpha_o} = U_{\alpha_m}$. Hence, considering χ_g as an element of $\mathrm{Hom}(\pi_1(\mathfrak{U}, U_o), G)/G$, that element is independent of the choice of cocycle $(\varphi_{\alpha\beta})$ representing the cohomology class φ; and the map $\varphi \longrightarrow \chi_\varphi$ thus takes $H^1(\mathfrak{U}, G)$ into $\mathrm{Hom}(\pi_1(\mathfrak{U}, U_o), G)/G$. We shall actually show that this mapping is a one-to-one correspondence. First, suppose that $(\varphi_{\alpha\beta})$ and $(\psi_{\alpha\beta})$ are two cocycles in $Z^1(\mathfrak{U}, G)$ leading to the same element in $\mathrm{Hom}(\pi_1(\mathfrak{U}, U_o), G)$; recalling (15), it is clear that $\psi_{\alpha\beta}$ can be replaced by an equivalent cocycle such that

$$\psi(U_{\alpha_0},U_{\alpha_1},\ldots,U_{\alpha_m}) = \varphi(U_{\alpha_0},U_{\alpha_1},\ldots,U_{\alpha_m})$$

for every closed chain. Then for any element $U_\alpha \in \mathcal{U}$ select a chain $\pi_\alpha = (U_{\alpha_0},U_{\alpha_1},\ldots,U_{\alpha_{m-1}},U_\alpha)$ and put $\theta_\alpha = \psi(\pi_\alpha^{-1})\varphi(\pi_\alpha)$, where π_α^{-1} indicates the chain π_α in the reverse order. The element θ_α is independent of the choice of chain π_α; for if $\tilde{\pi}_\alpha$ is another such chain, then

$\psi(\tilde{\pi}_\alpha^{-1})\varphi(\tilde{\pi}_\alpha) = \psi(\pi_\alpha^{-1})\psi(\pi_\alpha\tilde{\pi}_\alpha^{-1})\cdot\varphi(\tilde{\pi}_\alpha\pi_\alpha^{-1})\varphi(\pi_\alpha) = \psi(\pi_\alpha^{-1})\varphi(\pi_\alpha)$, since $\pi_\alpha\tilde{\pi}_\alpha^{-1}$ is a closed chain. Then $\psi_{\alpha\beta} = \psi(U_\alpha,U_\beta) =$
$= \psi(U_\alpha,U_{\alpha_{m-1}},\ldots,U_{\alpha_1},U_{\alpha_0},U_{\alpha_0},U_{\alpha_1},\ldots,U_{\alpha_{m-1}},U_\alpha,U_\beta) = \psi(\pi_\alpha^{-1})\psi(\pi_\beta) =$
$= \theta_\alpha\varphi(\pi_\alpha^{-1})\cdot\varphi(\pi_\beta)\theta_\beta^{-1} = \theta_\alpha\cdot\varphi_{\alpha\beta}\cdot\theta_\beta^{-1}$. The cocycles $(\varphi_{\alpha\beta})$ and $(\psi_{\alpha\beta})$ are hence equivalent, so the map $\varphi \longrightarrow \chi_\varphi$ is one-to-one into. Next, for any element in $\mathrm{Hom}(\pi_1(\mathcal{U},U_0),G)/G$, select a representative $\chi \in \mathrm{Hom}(\pi_1(\mathcal{U},U_0),G)$. Again for any element $U_\alpha \in \mathcal{U}$ select a chain $\pi_\alpha = (U_{\alpha_0},U_{\alpha_1},\ldots,U_{\alpha_{m-1}},U_\alpha)$; and whenever $U_\alpha \cap U_\beta \neq 0$ define $\varphi_{\alpha\beta} = \chi(\pi_\alpha,U_\alpha,U_\beta,\pi_\beta^{-1})$, noting that $(\pi_\alpha,U_\alpha,U_\beta,\pi_\beta^{-1})$ is a closed chain. Then whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq 0$, observe that $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \chi(\pi_\alpha,U_\alpha,U_\beta,\pi_\beta^{-1})\chi(\pi_\beta,U_\beta,U_\gamma,\pi_\gamma^{-1}) =$
$= \chi(\pi_\alpha,U_\alpha,U_\gamma,\pi_\gamma^{-1}) = \varphi_{\alpha\gamma}$, since the chains $(\pi_\alpha,U_\alpha,U_\beta,\pi_\beta^{-1},\pi_\beta,U_\beta,U_\gamma,\pi_\gamma^{-1})$ and $(\pi_\alpha,U_\alpha,U_\gamma,\pi_\gamma^{-1})$ are homotopic. Thus $(\varphi_{\alpha\beta})$ determines a cocycle in $Z^1(\mathcal{U},G)$; and it is clear that χ is the element associated to the cohomology class of φ under the correspondence considered here, thus showing that the map $\varphi \longrightarrow \chi_\varphi$ is onto as well.

Thus for any covering $\mathcal{U}$ with a selected base point $U_0 \in \mathcal{U}$ there has been defined a natural one-to-one correspondence $H^1(\mathcal{U},G) \longleftrightarrow \mathrm{Hom}(\pi_1(\mathcal{U},U_0),G)/G$. It is evident that these correspondences behave suitably under refinements of the covering, and hence in the limit lead to the desired correspondence; details again will be left to the reader.

Remark. The elements of $H^1(M,G)$ can be viewed as fibre bundles with totally disconnected group, (flat fibre bundles); and Lemma 27 is just the classification theorem for such bundles in terms of their characteristic classes. For a discussion from this point of view, see N. E. Steenrod, The Topology of Fibre Bundles, §13, (Princeton University Press, 1951).

At this stage it would perhaps be of interest to see some examples. It should be remarked that the group $\pi_1(M,p)$ defined above is actually isomorphic to the fundamental group of the surface, hence will be assumed known to the reader. (See the topological appendix for some further discussion.) First, as a rather trivial example, the projective line $\mathbb{P}$ has a trivial fundamental group, $\pi_1(\mathbb{P}) = 1$. For any group G it then follows from Lemma 27 that $H^1(M,G)$ consists of a single element; and considering in particular the group $G = PL(2,\mathbb{C})$, there is by Theorem 20 a unique projective structure on $\mathbb{P}$, which should not be very surprising. Next, for a compact surface of genus 1 the fundamental group is a free abelian group on two generators: $\pi_1(M) \cong \mathbb{Z} + \mathbb{Z}$. Letting π_1, π_2 be two generators, an element $X \in Hom(\pi_1(M),G)$ is completely determined by the values $X_i = X(\pi_i)$, and these are arbitrary subject only to the condition that $X_1X_2 = X_2X_1$; thus we can identify $Hom(\pi_1(M),G) = \{(X_1,X_2) \in G \times G \mid X_1X_2 = X_2X_1\}$. Calling two pairs (X_1,X_2) and (X_1',X_2') equivalent if for some element $g \in G$ they satisfy $X_i' = g^{-1}X_i g$ for $i = 1,2$, the set of these equivalence classes can be identified with $\mathrm{Hom}(\pi_1(M),G)/G$.

For the special case $G = PL(2,\mathbf{C})$, recall from the exact sequence (9) that an element $X \in G$ can be represented uniquely up to sign by a matrix $T \in SL(2,\mathbf{C})$. Recall also that by an inner automorphism any element $T \in SL(2,\mathbf{C})$ can be reduced to one of the following canonical forms:

$$(16) \quad \begin{array}{l} \text{(i)} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad \text{(ii)} \quad T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} , \ b \neq 0 ; \\ \text{(iii)} \quad T = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} , \quad a \neq 0, \pm 1 ; \end{array}$$

and except for replacing a by $1/a$ in (iii), none of these matrices is equivalent to any other under an inner automorphism. Now consider a pair of elements (X_1, X_2), $X_i \in PL(2,\mathbf{C})$, such that $X_1X_2 = X_2X_1$; if T_i is a matrix in $SL(2,\mathbf{C})$ representing X_i , then $T_1T_2 = \pm T_2T_1$. By an inner automorphism, T_1 can be reduced to one of the forms (16); thus there are three cases to consider. (i) If T_1 is the identity, then T_2 is arbitrary; and by a further inner automorphism, T_2 can also be reduced to one of the standard forms (16). (ii) If T_1 has the form (ii) and $T_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, then T_2 must satisfy the condition

$$\begin{pmatrix} a_2+b_1c_2 & b_2+b_1d_2 \\ c_2 & d_2 \end{pmatrix} = \pm \begin{pmatrix} a_2 & a_2b_1+b_2 \\ c_2 & c_2b_1+d_2 \end{pmatrix} .$$

If the $+$ sign holds, it is easy to see that $c_2 = 0$ and $a_2 = d_2 = \pm 1$; while the $-$ sign can clearly never occur.

The forms of T_1 and T_2 cannot be further changed by an inner automorphism. (iii) If T_1 has the form (iii), then T_2 must satisfy the condition

$$\begin{pmatrix} a_2a_1 & b_2a_1 \\ c_2/a_1 & d_2/a_1 \end{pmatrix} = \pm \begin{pmatrix} a_2a_1 & b_2/a_1 \\ c_2a_1 & d_2/a_1 \end{pmatrix} .$$

If the + sign holds, it is easy to see that $b_2 = c_2 = 0$, and $a_2 = 1/d_2$; further inner automorphisms can only have the effect of simultaneously replacing a_1, a_2 by $1/a_1$, $1/a_2$. If the − sign holds, it follows readily that $a_1 = \pm i$, $a_2 = d_2 = 0$, and $c_2 = -1/b_2$; by an inner automorphism, it is further possible to make $b_2 = +1$. Now considering all three cases together, and writing the elements as projective transformations, the elements of $H^1(M, PL(2,\mathbf{C})) \cong \mathrm{Hom}(\pi_1(M), PL(2,\mathbf{C}))/PL(2,\mathbf{C})$ can be represented by the distinct pairs of transformations on the following list:

$$(17) \begin{cases} \text{(i)} \quad T_1z = a_1z + b_1, \quad T_2z = a_2z + b_2 , \\ \qquad\qquad \text{where } a_1a_2 \neq 0, \quad (a_1-1)b_1 = (a_2-1)b_2 = 0 ; \\ \text{(ii)} \quad T_1z = -z , \qquad T_2z = -1/z . \end{cases}$$

These pairs of transformations are inequivalent, with the exceptions that

$$T_1z = a_1z, \quad T_2z = a_2z \quad \text{and} \quad T_1z = \frac{1}{a_1}z, \quad T_2z = \frac{1}{a_2}z$$

are equivalent.

Two aspects of this description should be pointed out. First, the image $\mu^*H^1(M, SL(2,\mathbf{C})) \subset H^1(M, PL(2,\mathbf{C}))$ under the mapping (10) consists precisely of the elements representable in the form (17 (i)); for only in the case (17 (ii)) was the matrix equation $T_1T_2 = - T_2T_1$ solvable. This example thus shows that the first assertion in Lemma 26 is a non-vacuous one. By that lemma, the element (17 (ii)) can never be the coordinate cohomology class of a projective structure. Second, all the elements (17 (i)) are actually affine, hence the cohomology classes so represented can be reduced to affine cohomology classes. Thus, recalling the first comment above and applying Theorem 20, it follows that any projective structure on a compact Riemann surface of genus one can be reduced to an affine structure; in other words, projective and affine structures coincide in this case.

(c) The main problem here is that of determining the projective structures on a compact Riemann surface in a sufficiently explicit manner; in view of Theorem 20 and the subsequent discussion, this problem can be rephrased as that of determining explicitly which elements of $H^1(M,PL(2,\mathbb{C})) \cong \text{Hom}(\pi_1(M),G)/G$ are the coordinate cohomology classes of projective structures on the Riemann surface M. Although some further discussion of properties of complex vector bundles seems necessary before directly tackling this problem, there are some interesting preliminary results which sould be mentioned here. The projective structures on M correspond to projective connections, as in Theorem 19; so the question arises, how to determine the coordinate cohomology class of the projective structure corresponding to a given projective connection on M.

To begin, select a complex line bundle $\lambda \in H^1(M, \mathcal{O}^*)$ such that $\lambda^2 = \kappa$, where κ is the canonical bundle of the Riemann surface. Since the Chern class of κ is even, and the group of line bundles of Chern class zero has the simple form described in §8, it is evident that there exists such a line bundle. There is not a unique such bundle of course, and for present purposes any choice will suffice. It should be noted that these bundles λ can be described very simply in terms of any projective structure on the Riemann surface M. As in Lemma 26, choose a cohomology class $T \in H^1(M, SL(2,\mathbb{C}))$ representing that projective structure; if $\{U_\alpha, z_\alpha\}$ is a projective coordinate covering for the structure, and $(T_{\alpha\beta}) \in Z^1(\mathcal{U}, SL(2,\mathbb{C}))$ are matrices representing the cohomology class T, then the coordinate functions (z_α) satisfy equation (11). It follows readily that the functions

$$\lambda_{\alpha\beta}(p) = c_{\alpha\beta} z_\beta(p) + d_{\alpha\beta}, \qquad p \in U_\alpha \cap U_\beta, \tag{18}$$

represent a complex line bundle $\lambda \in H^1(M, \mathcal{O}^*)$ for which $\lambda^2 = \kappa$, as desired.

<u>Lemma 28.</u> Let M be a compact Riemann surface, $h = (h_\alpha) \in C^0(\mathcal{U}, \mathcal{O}(\kappa^2))$ be a projective connection on the surface, and $\lambda \in H^1(M, \mathcal{O}^*)$ be a complex line bundle such that $\lambda^2 = \kappa$ is the canonical bundle of the surface. In each coordinate neighborhood U_α of a coordinate covering of the surface select two linearly independent analytic functions $f_{1\alpha}(z_\alpha)$, $f_{2\alpha}(z_\alpha)$ which are solutions of the differential equation

(19) $$2f''_\alpha(z_\alpha) + h_\alpha(z_\alpha)f_\alpha(z_\alpha) = 0 \;;$$

and introduce the vector-valued functions $\underline{f}_\alpha(z_\alpha) = \begin{pmatrix} f_{1\alpha}(z_\alpha) \\ f_{2\alpha}(z_\alpha) \end{pmatrix}$.
Then to each intersection $U_\alpha \cap U_\beta$ there corresponds a unique matrix $T_{\alpha\beta} \in GL(2,\mathbf{C})$ such that

(20) $$\underline{f}_\alpha(z_\alpha(p)) = \lambda_{\alpha\beta}(p)^{-1}T_{\alpha\beta}\underline{f}_\beta(z_\beta(p)) \text{ for } p \in U_\alpha \cap U_\beta \,.$$

These matrices form a cocycle $T \in Z^1(\mathfrak{U}, GL(2,\mathbf{C}))$ such that $\mu^*(T) \in H^1(M, PL(2,\mathbf{C}))$ is the coordinate cohomology class of the projective structure on M corresponding to the projective connection h .

Proof. Note that (19) is a linear differential equation with complex analytic coefficients, and the coefficient of the highest term is nowhere zero. Hence, as is well known, there exist complex analytic solutions in a small enough neighborhood U_α of any point on the surface; and the set of all these solutions form a two-dimensional complex vector space. If $f_\alpha(z_\alpha)$ is any solution in the open set $V_\alpha = z_\alpha(U_\alpha) \subset \mathbf{C}$, and if $U_\alpha \cap U_\beta \neq \emptyset$, then introduce the complex analytic function $g_\beta(z_\beta)$ in $z_\beta(U_\alpha \cap U_\beta) \subset V_\beta$ defined by

$$g_\beta(z_\beta(p)) = \lambda_{\alpha\beta}(p)f_\alpha(z_\alpha(p)) \,, \qquad \text{for } p \in U_\alpha \cap U_\beta \,.$$

Applying the chain rule for differentiation,

$$\begin{aligned} g'_\beta(z_\beta) &= \frac{d}{dz_\beta}\,[\lambda_{\alpha\beta}(z_\beta)f_\alpha(z_\alpha)] \\ &= \lambda'_{\alpha\beta}(z_\beta)f_\alpha(z_\alpha) + \lambda_{\alpha\beta}(z_\beta)^{-1}f'_\alpha(z_\alpha) \,, \end{aligned}$$

since $dz_\alpha/dz_\beta = \kappa_{\alpha\beta}(z_\beta)^{-1} = \lambda_{\alpha\beta}(z_\beta)^{-2}$. Differentiating again,

$$g''_\beta(z_\beta) = \lambda''_{\alpha\beta}(z_\beta)f_\alpha(z_\alpha) + \lambda_{\alpha\beta}(z_\beta)^{-3}f''_\alpha(z_\alpha) .$$

Since $f_\alpha(z_\alpha)$ is by assumption a solution of the differential equation (19), it follows that

$$\begin{aligned} g''_\beta(z_\beta) &= \lambda''_{\alpha\beta}(z_\beta)f_\alpha(z_\alpha) - \tfrac{1}{2}\, h_\alpha(z_\alpha)\lambda_{\alpha\beta}(z_\beta)^{-3}f_\alpha(z_\alpha) \\ &= [\lambda_{\alpha\beta}(z_\beta)^{-1}\lambda''_{\alpha\beta}(z_\beta) - \tfrac{1}{2}\, h_\alpha(z_\alpha)\lambda_{\alpha\beta}(z_\beta)^{-4}]g_\beta(z_\beta) . \end{aligned}$$

For the coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$ it follows readily from formula (4) that

$$\sigma_{2\alpha\beta}(z_\beta) = \theta_2 f_{\alpha\beta}(z_\beta) = -4\,\lambda_{\alpha\beta}(z_\beta)^{-1}\lambda''_{\alpha\beta}(z_\beta) ;$$

and recalling the defining property of a projective connection as given in formula (7),

$$h_\beta(z_\beta) = \lambda_{\alpha\beta}(z_\beta)^{-4}h_\alpha(z_\alpha) - 4\,\lambda_{\alpha\beta}(z_\beta)^{-1}\lambda''_{\alpha\beta}(z_\beta) .$$

Consequently $g''_\beta(z_\beta) = -\frac{1}{4}\, h_\beta(z_\beta)g_\beta(z_\beta)$; that is to say, the function $g_\beta(z_\beta)$ is a solution of the differential equation (19) in the set $z_\beta(U_\alpha \cap U_\beta) \subset V_\beta$. Now if $f_{1\alpha}(z_\alpha)$ and $f_{2\alpha}(z_\alpha)$ are two linearly independent solutions of (19) in the set $z_\alpha(U_\alpha)$, the functions $g_{1\beta}(z_\beta) = \lambda_{\alpha\beta}f_{1\alpha}(z_\alpha)$ and $g_{2\beta}(z_\beta) = \lambda_{\alpha\beta}f_{2\alpha}(z_\alpha)$ are linearly independent solutions of (19) in the set $z_\beta(U_\alpha \cap U_\beta)$; so for any other linearly independent solutions $f_{1\beta}(z_\beta)$, $f_{2\beta}(z_\beta)$ of (19) in $z_\beta(U_\beta)$, the functions $g_{i\beta}(z_\beta)$ are unique linear combinations of the functions $f_{i\beta}(z_\beta)$. This demonstrates equation (20), and at the same time shows that the matrices $(T_{\alpha\beta})$ form a cocycle $T \in Z^1(\mathfrak{U}, GL(2,\mathbb{C}))$.

. . . the meromorphic functions $w_\alpha(z_\alpha) = f_{1\alpha}(z_\alpha)/f_{2\alpha}(z_\alpha)$ in each coordinate neighborhood $z_\alpha(U_\alpha) \subset \mathbb{C}$. Since the functions $f_{i\alpha}(z_\alpha)$ are linearly independent solutions of (19), their Wronskian $f'_{1\alpha}(z_\alpha)f_{2\alpha}(z_\alpha) - f_{1\alpha}(z_\alpha)f'_{2\alpha}(z_\alpha)$ is nowhere vanishing in $z_\alpha(U_\alpha)$; but this clearly means that $w'_\alpha(z_\alpha) \neq 0$ at the regular points of that function, and $(1/w_\alpha(z_\alpha))' \neq 0$ at the poles of the function $w_\alpha(z_\alpha)$. Thus the mapping $z_\alpha \longrightarrow w_\alpha(z_\alpha)$ is a local homeomorphism from $z_\alpha(U_\alpha)$ into $\mathbb{P}$, and the composite functions $w_\alpha \circ z_\alpha$ are a complex analytic coordinate mapping on the Riemann surface M. Differentiating the function w_α, note that

$$w'_\alpha = w_\alpha\left(\frac{f'_{1\alpha}}{f_{1\alpha}} - \frac{f'_{2\alpha}}{f_{2\alpha}}\right) .$$

Since the functions $f_{i\alpha}$ are solutions of (19),

$$\left(\frac{f'_{i\alpha}}{f_{i\alpha}}\right) = -\frac{1}{2}h_\alpha - \left(\frac{f'_{i\alpha}}{f_{i\alpha}}\right)^2 ;$$

therefore

$$w''_\alpha = w_\alpha\left\{-\left(\frac{f'_{1\alpha}}{f_{1\alpha}}\right)^2 + \left(\frac{f'_{2\alpha}}{f_{2\alpha}}\right)^2\right\} + w_\alpha\left\{\frac{f'_{1\alpha}}{f_{1\alpha}} - \frac{f'_{2\alpha}}{f_{2\alpha}}\right\}^2$$

$$= -2\,w'_\alpha\,\frac{f'_{2\alpha}}{f_{2\alpha}} .$$

Differentiating once again,

$$w'''_\alpha = h_\alpha w'_\alpha + 6\,w'_\alpha\left(\frac{f'_{2\alpha}}{f_{2\alpha}}\right)^2 .$$

Finally, substituting these results into (2), it follows that

$$\theta_2 w_\alpha(z_\alpha) = h_\alpha(z_\alpha) .$$

Recalling Theorem 19, this means that the coordinate mappings $w_\alpha \circ z_\alpha$ are a projective coordinate covering representing the projective structure corresponding to the connection. The coordinate cohomology class is therefore defined by the coordinate transition functions for this coordinate covering; and by (20), this class is just $\mu^*(T)$, where $\mu^*: H^1(M, GL(2,\mathbb{C})) \to H^1(M, PL(2,\mathbb{C}))$ is the mapping associated to the natural mapping $\mu: GL(2,\mathbb{C}) \to PL(2,\mathbb{C})$. This then completes the proof.

Again let $\lambda \in H^1(M, \mathcal{O}^*)$ be any complex line bundle such that $\lambda^2 = \kappa$ is the canonical bundle on M, and let $(\lambda_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ be a representative cocycle. Then consider the holomorphic non-singular matrix valued functions $\Lambda_{\alpha\beta}$ defined in intersections $U_\alpha \cap U_\beta$ by

$$(21) \qquad \Lambda_{\alpha\beta}(z_\beta) = \begin{pmatrix} \lambda_{\alpha\beta}(z_\beta) & \frac{d}{dz_\beta}\lambda_{\alpha\beta}(z_\beta) \\ 0 & \lambda_{\alpha\beta}(z_\beta)^{-1} \end{pmatrix} \quad \text{for} \quad z_\beta \in z_\beta(U_\alpha \cap U_\beta).$$

It is a straightforward matter to verify that

$$(22) \qquad \Lambda_{\alpha\beta}(z_\beta(p)) \cdot \Lambda_{\beta\gamma}(z_\gamma(p)) = \Lambda_{\alpha\gamma}(z_\gamma(p)) \quad \text{if} \quad p \in U_\alpha \cap U_\beta \cap U_\gamma ;$$

the details will be left to the reader. Also, let $\mu^*: H^1(M, GL(2,\mathbb{C})) \to H^1(M, PL(2,\mathbb{C}))$ be induced by the natural homomorphism $\mu: GL(2,\mathbb{C}) \to PL(2,\mathbb{C})$.

Theorem 21. A cohomology class $T \in H^1(M, GL(2,\mathbb{C}))$ on a compact Riemann surface of genus $g > 1$ has the property that $\mu^* T \in H^1(M, PL(2,\mathbb{C}))$ is the coordinate cohomology class of a projective structure on the Riemann surface if and only if, for a

... coordinate covering $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ of M and a representative cocycle $(T_{\alpha\beta}) \in Z^1(\mathfrak{U}, GL(2,\mathbb{C}))$, there exist holomorphic non-singular matrix-valued functions F_α in the various sets U_α such that

$$F_\alpha(p)\Lambda_{\alpha\beta}(p) = T_{\alpha\beta}F_\beta(p) \quad \text{for} \quad p \in U_\alpha \cap U_\beta .$$

Proof. If $\mu^* T$ is the coordinate cohomology class of a projective structure on the Riemann surface then by Lemma 28 there exist holomorphic vector-valued functions $\underline{f}_\alpha$ in the various sets U_α, such that the entries in $\underline{f}_\alpha$ are linearly independent solutions of (19) in U_α, and that

(23) $$\underline{f}_\alpha(z_\alpha)\lambda_{\alpha\beta}(z_\beta) = T_{\alpha\beta}\underline{f}_\beta(z_\beta) \quad \text{in} \quad U_\alpha \cap U_\beta .$$

Differentiating the above equation with respect to z_β and recalling that $dz_\alpha/dz_\beta = \lambda_{\alpha\beta}(z_\beta)^{-2}$, secure that

(24) $$\underline{f}'_\alpha(z_\alpha)\lambda_{\alpha\beta}(z_\beta)^{-1} + \underline{f}_\alpha(z_\alpha)\lambda'_{\alpha\beta}(z_\beta) = T_{\alpha\beta}\underline{f}'_\beta(z_\beta) .$$

Introducing the holomorphic matrix-valued functions

$$F_\alpha(z_\alpha) = (\underline{f}_\alpha(z_\alpha), \underline{f}'_\alpha(z_\alpha)) ,$$

equations (23) and (24) can be written together in the form

$$F_\alpha(z_\alpha)\Lambda_{\alpha\beta}(z_\beta) = T_{\alpha\beta}F_\beta(z_\beta) ;$$

and since the entries of $\underline{f}_\alpha(z_\alpha)$ are linearly independent solutions of (19), their Wronskian $\det F_\alpha(z_\alpha)$ is nowhere vanishing. These functions $F_\alpha(z_\alpha)$ are then of the desired form.

On the other hand, suppose that there exist holomorphic matrix-valued functions F_α satisfying the conditions of the theorem, and let $\underline{f}_\alpha$ be the holomorphic vector-valued functions consisting of the first column of the matrix F_α. The functions $\underline{f}_\alpha(z_\alpha)$ satisfy equation (23) then, so their derivatives necessarily satisfy equation (24); and the holomorphic matrix-valued functions $G_\alpha(z_\alpha) = (\underline{f}_\alpha(z_\alpha), \underline{f}'_\alpha(z_\alpha))$ also satisfy $G_\alpha(z_\alpha)\Lambda_{\alpha\beta}(z_\beta) = T_{\alpha\beta}G_\beta(z_\beta)$. Therefore $\det G_\alpha(z_\alpha) = \det T_{\alpha\beta} \cdot \det G_\beta(z_\beta)$, or in other words, the functions $\det G_\alpha$ are holomorphic cross-sections of the complex line bundle $(\det T_{\alpha\beta})$; and since $T_{\alpha\beta}$ are constant matrices, so that $(\det T_{\alpha\beta})$ has Chern class zero, either $\det G_\alpha(z_\alpha) \equiv 0$ or $\det G_\alpha(z_\alpha)$ never vanishes. In the second case, it follows immediately that the mapping $z_\alpha \longrightarrow w_\alpha(z_\alpha)$, defined by $w_\alpha(z_\alpha) = f_{1\alpha}(z_\alpha)/f_{2\alpha}(z_\alpha)$, is a local homeomorphism; and from equation (23), the local coordinate functions $w_\alpha \circ z_\alpha$ define a projective structure on M, with coordinate cohomology class T. Thus to prove the theorem, it is only necessary to show that $\det G_\alpha(z_\alpha) \not\equiv 0$. Now if $\det G_\alpha(z_\alpha) \equiv 0$, the entries in the vector-valued functions $\underline{f}_\alpha(z_\alpha)$ would be linearly dependent; thus these functions could be written in the form $\underline{f}_\alpha(z_\alpha) = f_\alpha(z_\alpha)\underline{c}_\alpha$, where $f_\alpha(z_\alpha)$ are ordinary holomorphic functions and $\underline{c}_\alpha$ are constant vectors. Note that neither $f_\alpha(z_\alpha)$ nor $\underline{c}_\alpha$ can be zero anywhere, since the original function $F_\alpha(z_\alpha)$ was by assumption non-singular. Equation (21) would now have the form

$$f_\alpha(z_\alpha)\lambda_{\alpha\beta}(z_\beta)\underline{c}_\alpha = f_\beta(z_\beta)T_{\alpha\beta}\underline{c}_\beta ;$$

but this would imply that $f_\alpha(z_\alpha)\lambda_{\alpha\beta}(z_\beta) = c_{\alpha\beta}f_\beta(z_\beta)$ for some constants $c_{\alpha\beta}$. This latter condition is impossible; for since the functions $f_\alpha(z_\alpha)$ are nowhere vanishing, the line bundles $\lambda_{\alpha\beta}$ and $c_{\alpha\beta}$ would necessarily be equivalent, but their Chern classes are $g-1$ and 0 respectively. This contradiction completes the proof of the theorem.

Corollary. The functions $F_\alpha(z_\alpha)$ in Theorem 21 can be so chosen that

$$\frac{d}{dz_\alpha} F_\alpha(z_\alpha) = F_\alpha(z_\alpha) \begin{pmatrix} 0 & -\frac{1}{2} h_\alpha(z_\alpha) \\ 1 & 0 \end{pmatrix},$$

where $(h_\alpha(z_\alpha))$ is the projective connection corresponding to the projective structure.

Proof. The matrix function $F_\alpha(z_\alpha)$ can always be taken in the form $F_\alpha(z_\alpha) = (\underline{f}_\alpha(z_\alpha), \underline{f}'_\alpha(z_\alpha))$, where $\underline{f}_\alpha(z_\alpha)$ is a solution of the differential equation $2\underline{f}''_\alpha(z_\alpha) + h_\alpha(z_\alpha)\underline{f}_\alpha(z_\alpha) = 0$, as in Lemma 28. Then

$$\begin{aligned}\frac{d}{dz_\alpha} F_\alpha(z_\alpha) &= (\underline{f}'_\alpha(z_\alpha), \underline{f}''_\alpha(z_\alpha)) = (\underline{f}'_\alpha(z_\alpha), -\frac{1}{2} h_\alpha(z_\alpha)\underline{f}_\alpha(z_\alpha)) \\ &= (\underline{f}_\alpha(z_\alpha), \underline{f}'_\alpha(z_\alpha)) \begin{pmatrix} 0 & -\frac{1}{2} h_\alpha(z_\alpha) \\ 1 & 0 \end{pmatrix},\end{aligned}$$

as desired.

Let $\mathcal{G}_2$ be the sheaf of germs of complex analytic mappings from a Riemann surface M into the group $GL(2,\mathbb{C})$; note that this is merely a sheaf of sets, with no group structure defined. As in the discussion on page 174 and the following pages, it is possible to introduce a non-abelian cohomology set $H^1(M, \mathcal{G}_2)$; the elements of this cohomology set are called <u>complex vector bundles</u>

of rank 2 over the Riemann surface M. The concept is straightforward, in view of what has been discussed so far, but a more detailed treatment will be postponed to a later section. The constant functions form a subsheaf $GL(2,\mathbb{C}) \subset \mathcal{B}_2$, and the inclusion mapping between these sheaves induces a mapping $i^*: H^1(M,GL(2,\mathbb{C})) \longrightarrow H^1(M,\mathcal{B}_2)$; thus the flat complex vector bundles introduced on page 178 determine well-defined complex vector bundles. In this terminology, Theorem 21 asserts that the flat vector bundles $T \in H^1(M,GL(2,\mathbb{C}))$ such that $\mu^*T \in H^1(M,PL(2,\mathbb{C}))$ is the coordinate cohomology class of a projective structure on the Riemann surface are precisely those flat vector bundles such that $i^*T = \Lambda \in H^1(M, \mathcal{B}_2)$, where Λ is the complex vector bundle defined by the transition functions (21). The problem is that of determining explicitly the flat forms of that canonical complex vector bundle.

(d) On a surface endowed with a projective structure, there is a generalization of the deRham sequence which is of some interest; this was introduced by M. Eichler, who used it in studying algebraic correspondences on some Riemann surfaces arising in number theory. (See M. Eichler, "Eine Verallgemeinerung der Abelschen Integrale," Math. Zeitschr. 67(1957), 267-298; and G. Shimura, "Sur les intégrales attachées aux formes automorphes," Jour. Math. Soc. of Japan 11(1959), 291-311.) As yet this structure has not been studied extensively on general Riemann surfaces.

Let M be a Riemann surface with a fixed projective structure, and let $\lambda \in H^1(M, \mathcal{O}^*)$ be a complex line bundle of Chern class $c(\lambda) = g-1$. Choose a projective coordinate covering $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ of the surface M, belonging to the given projective structure, and let $z_\alpha = f_{\alpha\beta}(z_\beta)$ be the coordinate transition functions; thus $(f_{\alpha\beta}) \in Z^1(\mathfrak{U}, PL(2,\mathbb{C}))$ represents the coordinate cohomology class of that projective structure. As in Lemma 26, choose matrices $(T_{\alpha\beta}) \in Z^1(\mathfrak{U}, SL(2,\mathbb{C}))$ which represent these projective transformations, that is, such that $\mu^*(T_{\alpha\beta}) = f_{\alpha\beta}$. Writing these matrices out explicitly as

$$(25) \qquad T_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix} ,$$

recall as above that the functions $(c_{\alpha\beta} z_\beta + d_{\alpha\beta})$ represent a complex line bundle of Chern class $g-1$; therefore it is evident that

$$(26) \qquad \lambda_{\alpha\beta}(p) = \xi_{\alpha\beta}(c_{\alpha\beta} z_\beta(p) + d_{\alpha\beta}) \quad \text{for} \quad p \in U_\alpha \cap U_\beta ,$$

where $(\xi_{\alpha\beta})$ are complex constants representing a complex line bundle $\xi \in H^1(M, \mathcal{O}^*)$ of zero Chern class.

For any integer $n \geqq 0$ consider the sheaf $\mathcal{O}(\lambda^{-n})$ of germs of holomorphic cross-sections of the complex line bundle λ^{-n}. If $p \in U_\alpha$ is a point on the surface and $f \in \mathcal{O}(\lambda^{-n})_p$ is a germ of a cross-section at p, then f is represented by an analytic function $f_\alpha(z_\alpha)$ in some neighborhood of p; and if $p \in U_\alpha \cap U_\beta$, these local representations are related by

$$(27) \quad f_\beta(z_\beta) = \lambda_{\alpha\beta}(z_\beta)^n f_\alpha(z_\alpha) = \xi_{\alpha\beta}^n (c_{\alpha\beta} z_\beta + d_{\alpha\beta})^n f_\alpha\left(\frac{a_{\alpha\beta} z_\beta + b_{\alpha\beta}}{c_{\alpha\beta} z_\beta + d_{\alpha\beta}}\right)$$

Having written these conditions out so explicitly, there is an obvious subsheaf of $\mathcal{O}(\lambda^{-n})$, as follows. Note that if $f_\alpha(z_\alpha)$ is a polynomial in z_α of degree $\leq n$, then from (27) it necessarily follows that $f_\beta(z_\beta)$ is a polynomial in z_β of degree $\leq n$ also. Therefore we can introduce the subsheaf $\mathcal{P}_n(\lambda^{-n}) \subset \mathcal{O}(\lambda^{-n})$ consisting of those germs of analytic sections of λ^{-n} which are polynomials of degree $\leq n$ in any local projective coordinate system of the given projective structure. For the case $n = 0$, $\mathcal{P}_0(\lambda^0) = \mathbb{C}$, the subsheaf of constant functions; and this subsheaf is the kernel of the sheaf homomorphism $d: \mathcal{O} \longrightarrow \mathcal{O}(\kappa)$ defined by exterior differentiation. As a generalization:

<u>Lemma 29.</u> On any Riemann surface M with a fixed projective structure, there is an exact sequence of sheaves

$$(28) \quad 0 \longrightarrow \mathcal{P}_n(\lambda^{-n}) \xrightarrow{i} \mathcal{O}(\lambda^{-n}) \xrightarrow{d^{n+1}} \mathcal{O}(\xi^{-2n-2}\lambda^{n+2}) \longrightarrow 0$$

for any $n \geqq 0$, where d^{n+1} is the homomorphism which associates to a germ $f \in \mathcal{O}(\lambda^{-n})_p$ represented by a local analytic function $f_\alpha(z_\alpha)$ the germ $d^{n+1}f \in \mathcal{O}(\xi^{-2n-2}\lambda^{n+2})$ represented by the local analytic function $d^{n+1}f_\alpha(z_\alpha)/dz_\alpha^{n+1}$.

Proof. The first thing to prove is that d^{n+1} is a well-defined sheaf homomorphism. If $f \in \mathcal{O}(\lambda^{-n})_p$ is represented by two local analytic functions $f_\alpha(z_\alpha)$ and $f_\beta(z_\beta)$ when $p \in U_\alpha \cap U_\beta$, those two functions are related by formula (27). Select a simple closed curve $\gamma_\beta \subset z_\beta(U_\beta)$ encircling the point $z_\beta(p)$ and such that $f_\beta(z_\beta)$ is analytic on an open neighborhood form

$$(29)\quad 0 \to \Gamma(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n})) \xrightarrow{\delta^*} H^1(M, \mathcal{P}_n(\lambda^{-n})) \xrightarrow{\nu^*} \Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n}))^* \to 0,$$

where $\Gamma(-)^*$ denotes the dual vector space to $\Gamma(-)$.

Proof. The complex line bundle λ can be defined by a cocycle of the form (26) for some flat line bundle ξ , and $\eta = \xi^2$. The exact sequence (28) of Lemma 29 can be rewritten

$$0 \longrightarrow \mathcal{P}_n(\lambda^{-n}) \xrightarrow{i} \mathcal{O}(\lambda^{-n}) \xrightarrow{d^{n+1}} \mathcal{O}(\kappa^{n+1}\lambda^{-n}) \longrightarrow 0 .$$

The corresponding exact cohomology sequence on the surface M then begins

$$0 \longrightarrow \Gamma(M, \mathcal{P}_n(\lambda^{-n})) \xrightarrow{i^*} \Gamma(M, \mathcal{O}(\lambda^{-n})) \xrightarrow{(d^{n+1})^*} \Gamma(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n})) \xrightarrow{\delta^*}$$

$$\longrightarrow H^1(M, \mathcal{P}_n(\lambda^{-n})) \xrightarrow{i^*} H^1(M, \mathcal{O}(\lambda^{-n})) \xrightarrow{(d^{n+1})^*} H^1(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n})) \to \cdots$$

Since the line bundle λ^{-n} has Chern class $c(\lambda^{-n}) = -n(g-1) < 0$, it follows from the Corollary to Theorem 11 that $\Gamma(M, \mathcal{O}(\lambda^{-n})) = 0$; then of course $\Gamma(M, \mathcal{P}_n(\lambda^{-n})) = 0$ as well. From the Serre duality theorem, $H^1(M, \mathcal{O}(\lambda^{-n})) \cong \Gamma(M, \mathcal{O}(\kappa\lambda^n))^* = \Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n}))^*$; and $H^1(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n})) \cong \Gamma(M, \mathcal{O}(\kappa^{-n}\lambda^n))^* = \Gamma(M, \mathcal{O}(\eta^n\lambda^{-n}))^* = 0$, since $c(\eta^n\lambda^{-n}) = c(\lambda^{-n}) < 0$. Upon substituting these results in the exact cohomology sequence, there follows the exact sequence (29), as desired.

<u>Corollary.</u> If the complex line bundle λ in the preceding theorem is such that $\lambda^{2n} = \kappa^n$, then

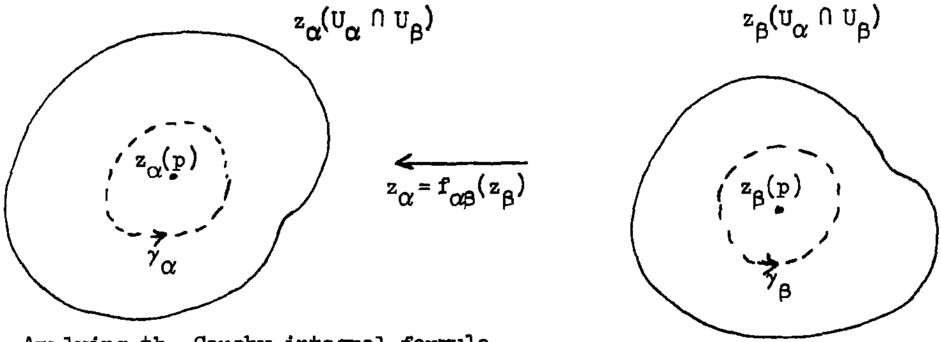
$$\dim H^1(M, \mathcal{P}_n(\lambda^{-n})) = 2 \dim \Gamma(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n})) .$$

Proof. The additional hypothesis is essentially just that $\eta^n = 1$; the first and last terms in the exact sequence (29) then have the same dimension, from which result the Corollary follows immediately.

The groups $H^1(M, \mathcal{P}_n(\lambda^{-n}))$ are called the <u>Eichler cohomology groups</u> associated to the given projective structure on the Riemann surface M . For the special case $n = 0$, note that $H^1(M, \mathcal{P}_0(\lambda^0)) = H^1(M,\mathbf{C})$, the ordinary cohomology of the surface M . For any value $\eta \geq 0$, these cohomology groups are finite-dimensional complex vector spaces, as a consequence of Theorem 21; and indeed, these groups can be described explicitly in a purely algebraic manner, in terms of the coordinate cohomology class of the given projective structure and the cohomology class of the flat line bundle η . Recalling from part (b) above that the coordinate cohomology class can be viewed as an element of $\operatorname{Hom}(\pi_1(M),PL(2,\mathbf{C}))/PL(2,\mathbf{C})$, and that the bundle η can be described as an element of $\operatorname{Hom}(\pi_1(M),\mathbf{C}^*)$, it is evident that the description involves only the fundamental group of the surface M. There is at present no need to carry out this description in great detail, so we shall consider the matter only rather briefly. An element $\sigma \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$ can be represented in terms of a projective covering $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ by a cocycle $\sigma_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{P}_n(\lambda^{-n}))$; each element $\sigma_{\alpha\beta}(z_\beta)$ is a polynomial in the variable z_β of degree at most n , and the cocycle condition is that

$$\sigma_{\alpha\gamma}(z_\gamma) = \lambda_{\beta\gamma}(z_\gamma)^n \, \sigma_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) + \sigma_{\beta\gamma}(z_\gamma) \tag{30}$$

of the closure of the interior of γ_β ; and suppose that γ_β is selected sufficiently near $z_\beta(p)$ that the function $f_\alpha(z_\alpha)$ is analytic on an open neighborhood of the closure of the interior of $\gamma_\alpha = f_{\alpha\beta}(\gamma_\beta) \subset z_\alpha(U_\alpha)$, where $z_\alpha = f_{\alpha\beta}(z_\beta)$ is the coordinate transition function.



Applying the Cauchy integral formula,

$$\frac{d^{n+1} f_\alpha(z_\alpha)}{dz_\alpha^{n+1}} = \frac{(n+1)!}{2\pi i} \int_{\zeta_\alpha \in \gamma_\alpha} \frac{f_\alpha(\zeta_\alpha)}{(\zeta_\alpha - z_\alpha)^{n+2}}\, d\zeta_\alpha \,.$$

Now make the changes of variables $z_\alpha = f_{\alpha\beta}(z_\beta)$, $\zeta_\alpha = f_{\alpha\beta}(\zeta_\beta)$, where $f_{\alpha\beta}$ is the projective transformation represented by the unimodular matrix (25). Note that

$$\zeta_\alpha - z_\alpha = \frac{a_{\alpha\beta}\zeta_\beta + b_{\alpha\beta}}{c_{\alpha\beta}\zeta_\beta + d_{\alpha\beta}} - \frac{a_{\alpha\beta}z_\beta + b_{\alpha\beta}}{c_{\alpha\beta}z_\beta + d_{\alpha\beta}} = \frac{\zeta_\beta - z_\beta}{(c_{\alpha\beta}\zeta_\beta + d_{\alpha\beta})(c_{\alpha\beta}z_\beta + d_{\alpha\beta})}$$

$$= (\zeta_\beta - z_\beta)\xi_{\alpha\beta}^2\, \lambda_{\alpha\beta}(\zeta_\beta)^{-1}\lambda_{\alpha\beta}(z_\beta)^{-1} \quad ,$$

and that

$$d\zeta_\alpha = (c_{\alpha\beta}\zeta_\beta + d_{\alpha\beta})^{-2} d\zeta_\beta = \xi_{\alpha\beta}^2\, \lambda_{\alpha\beta}(\zeta_\beta)^{-2} d\zeta_\beta \ ;$$

and recalling (27), it follows that

$$\frac{d^{n+1}f_\alpha(z_\alpha)}{dz_\alpha^{n+1}} = \frac{(n+1)!}{2\pi i} \int_{\zeta_\beta \in \gamma_\beta} \xi_{\alpha\beta}^{-2n-2} \lambda_{\alpha\beta}(z_\beta)^{n+2} \frac{f_\beta(\zeta_\beta)}{(\zeta_\beta - z_\beta)^{n+2}} d\zeta_\beta$$

$$= \xi_{\alpha\beta}^{-2n-2} \lambda_{\alpha\beta}(z_\beta)^{n+2} \frac{d^{n+1}f_\beta(z_\beta)}{dz_\beta^{n+1}} .$$

Thus these derivatives represent the same element of $\mathcal{O}(\xi^{-2n-2}\lambda^{n+2})$, so that the mapping d^{n+1} is well defined.

The kernel of the homomorphism d^{n+1} is the subsheaf consisting of those germs of analytic functions $f_\alpha(z_\alpha)$ such that $d^{n+1}f_\alpha(z_\alpha)/dz_\alpha^{n+1} = 0$, hence is just the subsheaf $\mathcal{P}_n(\lambda^{-n}) \subset \mathcal{O}(\lambda^{-n})$ of polynomials of degree at most n. This shows the exactness of the sequence (28) at the first two places, where i is the inclusion mapping. To complete the proof of exactness, for any element $f \in \mathcal{O}(\xi^{-2n-2}\lambda^{n+2})_p$ select a representative analytic function $f_\alpha(z_\alpha)$; choosing any complex analytic function $g_\alpha(z_\alpha)$ in an open neighborhood of p such that $d^{n+1}g_\alpha(z_\alpha)/dz_\alpha^{n+1} = f_\alpha(z_\alpha)$, the function $g_\alpha(z_\alpha)$ represents an element $g \in \mathcal{O}(\lambda^{-n})$ for which $d^{n+1}g = f$, in view of the first part of the proof. Thus the mapping d^{n+1} in (28) is onto, and the proof is thereby concluded.

<u>Theorem 22.</u> Let M be a compact Riemann surface of genus $g > 1$ with a fixed projective structure, and let $\lambda \in H^1(M, \mathcal{O}^*)$ be a complex line bundle of Chern class $c(\lambda) = g-1$; thus there is a flat line bundle η such that $\lambda^2 = \eta\kappa$, where κ is the canonical bundle of the surface M. There is then for any integer $n \geq 0$ an exact sequence of complex vector spaces of the

whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, where $f_{\beta\gamma} \in PL(2,\mathbf{C})$ is the coordinate transition function. Note that (30) is a formal identity among various polynomials, and does not involve the point set $U_\alpha \cap U_\beta \cap U_\gamma$ explicitly. A cocycle $\sigma_{\alpha\beta}$ is a coboundary if there are polynomials $\tau_\alpha(z_\alpha)$ of degree at most n such that

$$(31) \qquad \sigma_{\alpha\beta}(z_\beta) = \tau_\beta(z_\beta) - \lambda_{\alpha\beta}(z_\beta)^n \, \tau_\alpha(f_{\alpha\beta}(z_\beta))$$

whenever $U_\alpha \cap U_\beta \neq \emptyset$; and the cohomology group is the quotient of the group of cocycles by the group of coboundaries.

It is perhaps of some interest to see the explicit forms of the homomorphisms in the exact sequence (29). First, in terms of a projective coordinate covering $\mathfrak{U} = \{U_\alpha, z_\alpha\}$, a section $f \in \Gamma(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n}))$ is given by analytic functions $f_\alpha(z_\alpha) \in \Gamma(z_\alpha(U_\alpha), \mathcal{O})$ such that $f_\alpha(z_\alpha) = \kappa_{\alpha\beta}(z_\beta)^{n+1}\lambda_{\alpha\beta}(z_\beta)^{-n} f_\beta(z_\beta)$ in $U_\alpha \cap U_\beta$. The mapping δ^* is the coboundary operator derived in the familiar manner from the exact sequence of sheaves (28). Thus in each set $z_\alpha(U_\alpha) \subset \mathbf{C}$ select a complex analytic function $F_\alpha(z_\alpha)$ such that $d^{n+1}F_\alpha(z_\alpha)/dz_\alpha^{n+1} = f_\alpha(z_\alpha)$; any $(n+1)$-fold indefinite integral of the function $f_\alpha(z_\alpha)$ will do. The cohomology class $\delta^* f \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$ is represented by the cocycle

$$(32) \qquad \sigma_{\alpha\beta}(z_\beta) = F_\beta(z_\beta) - \lambda_{\alpha\beta}(z_\beta)^n \, F_\alpha(z_\alpha) \, .$$

The functions $\sigma_{\alpha\beta}(z_\beta)$ are necessarily polynomials of degree $\leq n$ then; and replacing the functions $F_\alpha(z_\alpha)$ by $F_\alpha(z_\alpha) + \tau_\alpha(z_\alpha)$ for polynomials $\tau_\alpha(z_\alpha)$ of degree $\leq n$, the most general possible choice for these functions $F_\alpha(z_\alpha)$, replaces the cocycle (32) by a cohomologous cocycle, in view of (31). Next consider a cohomology class $\sigma \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$, represented by a cocycle $\sigma_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{P}_n(\lambda^{-n}))$; the element $\nu^*\sigma$ is a linear functional on the space $\Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n})$, described explicitly as in the discussion of the Serre duality theorem. Namely, select any zero cochain $g = (g_\alpha) \in C^0(\mathcal{U}, \mathcal{E}(\lambda^{-n}))$ such that $\delta g = \sigma$; that is, select C^∞ functions $g_\alpha(z_\alpha)$ in $z_\alpha(U_\alpha)$ such that $\sigma_{\alpha\beta}(z_\beta) = g_\beta(z_\beta) - \lambda_{\alpha\beta}(z_\beta)^n g_\alpha(f_{\alpha\beta}(z_\beta))$ in $z_\beta(U_\alpha \cap U_\beta)$. Note that $(\bar\partial g_\alpha) \in Z^0(\mathcal{U}, \mathcal{E}^{0,1}(\lambda^{-n})) = \Gamma(M, \mathcal{E}^{0,1}(\lambda^{-n}))$. For any element $f = (f_\alpha) \in \Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n})) = \Gamma(M, \mathcal{O}^{1,0}(\eta^n\kappa^n\lambda^{-n}))$ the form $\bar\partial g_\alpha \wedge f_\alpha dz_\alpha \in \Gamma(M, \mathcal{E}^{1,1}(\eta^n\kappa^n\lambda^{-2n})) = \Gamma(M, \mathcal{E}^{1,1})$; and then

$$(\nu^*\sigma)(f) = \iint_M \bar\partial g_\alpha \wedge f_\alpha dz_\alpha . \tag{33}$$

For the case $n = 0$, the image $\delta^* f \in H^1(M,\mathbb{C})$ of an element $f \in \Gamma(M, \mathcal{O}(\kappa)) = \Gamma(M, \mathcal{O}^{1,0})$ is essentially the set of periods of the differential form f ; and as above, for any $n \geqq 0$ the image $\delta^* f \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$ of an element $f \in \Gamma(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n}))$ can be viewed as a generalized set of periods of the element f, derived from an (n+1)-fold indefinite integral. For any cohomology class $\sigma \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$ the vanishing of the image $\nu^*\sigma$, or equivalently the vanishing of the integrals (33) for all elements $f \in \Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n}))$, is the necessary and sufficient condition that σ be the generalized periods attached to a section in $\Gamma(M, \mathcal{O}(\kappa^{n+1}\lambda^{-n}))$.

The Eichler cohomology group can also be described slightly differently. It follows almost immediately from Lemma 29 that, on a Riemann surface M with a fixed projective structure, for any integer $n \geqq 0$ there is an exact sequence of sheaves of the form

$$(34) \qquad 0 \longrightarrow \mathcal{P}_n(\lambda^{-n}) \xrightarrow{i} \mathcal{M}(\lambda^{-n}) \xrightarrow{d^{n+1}} \mathcal{M}(\xi^{-2n-2}\lambda^{n+2}) ,$$

where $\mathcal{M}$ denotes the sheaf of germs of meromorphic functions. The homomorphism d^{n+1} in (34) is not onto, since a general meromorphic function does not have single-valued meromorphic indefinite integrals of all orders. The image $d^{n+1}\mathcal{M}(\lambda^{-n})$ is called the <u>sheaf of germs of meromorphic sections of the second kind of the bundle</u> λ^{-n}, and will be denoted by $\mathcal{M}_o(\xi^{-2n-2}\lambda^{n+2})$. It is evident that, if z_α is a local coordinate mapping at a point $p \in M$ such that $z_\alpha(p) = 0$, then $f \in \mathcal{M}(\xi^{-2n-2}\lambda^{n+2})_p$ is in the subsheaf $\mathcal{M}_o(\xi^{-2n-2}\lambda^{n+2})_p$ precisely when the germ f is represented by a meromorphic function $f_\alpha(z_\alpha)$ with a Laurent expansion of the form

$$(35) \qquad f_\alpha(z_\alpha) = \sum_{\nu=n+2}^{r} a_{-\nu} z^{-\nu} + \sum_{\nu=0}^{\infty} a_\nu z^\nu ;$$

for these are just the meromorphic functions which can be written $f_\alpha(z_\alpha) = d^{n+1}g_\alpha(z_\alpha)/dz_\alpha^{n+1}$ for some meromorphic function $g_\alpha(z_\alpha)$. The condition that a meromorphic function $f_\alpha(z_\alpha)$ have a Laurent expansion of the form (35) is not invariant under an arbitrary non-singular analytic change of coordinates, considering $(f_\alpha(z_\alpha))$ as an element of the sheaf $\mathcal{M}(\xi^{-2n-2}\lambda^{n+2})$; so the sheaf $\mathcal{M}_o(\xi^{-2n-2}\lambda^{n+2})$ depends upon the choice of the projective structure on the Riemann surface. (The case $n = 0$ is exceptional here. For the elements $(f_\alpha(z_\alpha)) \in \mathcal{M}(\xi^{-2}\lambda^2)_p = \mathcal{M}(\kappa)_p = \mathcal{M}_p^{1,0}$ can be viewed as meromorphic differential forms on M, and condition (35) can be rephrased as the condition that the form have residue zero; this expresses the condition intrinsically, in terms of the complex structure alone. The terminology "meromorphic functions of the second kind" is motivated by the usual terminology in this special case.)

<u>Theorem 23.</u> Let M be a compact Riemann surface with a fixed projective structure, and let $\lambda \in H^1(M, \mathcal{O}^*)$ be a complex line bundle of Chern class $c(\lambda) = g-1$. Then for any integer $n \geqq 0$, the Eichler cohomology group can be written canonically

$$H^1(M, \mathcal{P}_n(\lambda^{-n})) \cong \frac{\Gamma(M, d^{n+1}\mathcal{M}(\lambda^{-n}))}{d^{n+1}\Gamma(M, \mathcal{M}(\lambda^{-n}))} .$$

Proof. Modifying (34) to yield the exact sequence of sheaves

$$0 \longrightarrow \mathcal{P}_n(\lambda^{-n}) \longrightarrow \mathcal{M}(\lambda^{-n}) \xrightarrow{d^{n+1}} d^{n+1}\mathcal{M}(\lambda^{-n}) \rightarrow 0 ,$$

where $d^{n+1}\mathcal{M}(\lambda^{-n}) = \mathcal{M}_0(\xi^{-2n-2}\lambda^{n+2}) \subset \mathcal{M}(\xi^{-2n-2}\lambda^{n+2})$, the associated exact cohomology sequence begins

$$0 \longrightarrow \Gamma(M, \mathcal{P}_n(\lambda^{-n})) \longrightarrow \Gamma(M, \mathcal{M}(\lambda^{-n})) \xrightarrow{d^{n+1}} \Gamma(M, d^{n+1}\mathcal{M}(\lambda^{-n})) \xrightarrow{\delta^*}$$

(36)

$$\longrightarrow H^1(M, \mathcal{P}_n(\lambda^{-n})) \longrightarrow H^1(M, \mathcal{M}(\lambda^{-n})) \longrightarrow \cdots .$$

Selecting any section $g \in \Gamma(M, \mathcal{M}^*(\lambda^n))$, the operation of multiplication by g defines a sheaf isomorphism $\mathcal{M}(\lambda^{-n}) \cong \mathcal{M}$; therefore, by Theorem 12, $H^1(M, \mathcal{M}(\lambda^{-n})) \cong H^1(M, \mathcal{M}) = 0$. The desired theorem is then an immediate consequence of the exact sequence (36), and the proof is thereby concluded.

<u>Corollary 1.</u> On any compact Riemann surface M it follows that

$$H^1(M, \mathbf{C}) \cong \Gamma(M, d\mathcal{M})/d\Gamma(M, \mathcal{M}) .$$

Proof. This is just the special case $n = 0$ of Theorem 23, recalling that in this case the sheaf $d\mathcal{M} \subset \mathcal{M}^{1,0}$ is defined intrinsically, independently of a choice of projective structure on the surface.

<u>Remarks.</u> In the Corollary, the space $\Gamma(M, d\mathcal{M})$ is the space of meromorphic differentials of the second kind on the Riemann surface M, that is to say, is the space of meromorphic differentials with zero residue at each point of the surface. The Theorem itself can be restated somewhat more precisely as the assertion that, when the genus $g > 1$, there is an exact sequence of the form

$$(37) \quad 0 \to \Gamma(M, \mathcal{M}(\lambda^{-n})) \xrightarrow{d^{n+1}} \Gamma(M, d^{n+1}\mathcal{M}(\lambda^{-n})) \xrightarrow{\delta^*} H^1(M, \mathcal{P}_n(\lambda^{-n})) \to 0 ;$$

for as in the proof of Theorem 22 it follows that $\Gamma(M, \mathcal{P}_n(\lambda^{-n})) = 0$, so that (37) follows directly from (36).

For some purposes interest lies not just in the Eichler cohomology group itself, but also in the splitting of that group given by the exact sequence (29). (Compare the discussion of §8(b).) Thus one is led to consider the form of that splitting when the Eichler cohomology group is represented as in Theorem 23. To be explicit, consider a cohomology class $\sigma \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$ which is represented by a section $h = (h_\alpha) \in \Gamma(M, d^{n+1}\mathcal{M}(\lambda^{-n}))$ under the isomorphism given in Theorem 23. It is clearly sufficient to describe the image of σ under the homomorphism ν^*

of (29), in terms of the section h; recall that $\nu^*\sigma$ is a linear functional on the vector space $\Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n}))$. For each set U_α of a suiteble projective coordinate covering there will be a meromorphic function $H_\alpha(z_\alpha)$ such thet $d^{n+1}H_\alpha(z_\alpha)/dz_\alpha^{n+1} = h_\alpha(z_\alpha)$; these functions can be viewed as an element $(H_\alpha) \in C^0(\mathfrak{U}, \mathcal{M}(\lambda^{-n}))$. For any section $f = (f_\alpha) \in \Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n}))$ the products $f_\alpha(z_\alpha)H_\alpha(z_\alpha)$ form an element $fH = (f_\alpha H_\alpha) \in C^0(\mathfrak{U}, \mathcal{M}(\eta^n\kappa^{n+1}\lambda^{-2n})) = C^0(\mathfrak{U}, \mathcal{M}(\kappa)) \cong C^0(\mathfrak{U}, \mathcal{M}^{1,0})$; the residue of this meromorphic differential form is well defined locally by the Cauchy integral formula, and its total residue on the surface will be denoted by $\mathcal{R}(fH)$.

Corollary 2. If M is a compact Riemann surface of genus $g > 1$ and $\sigma \in H^1(M, \mathcal{P}_n(\lambda^{-n}))$ is represented by $h = (h_\alpha) = (d^{n+1}H_\alpha) \in \Gamma(M, d^{n+1}\mathcal{M}(\lambda^{-n}))$ under the isomorphism of Theorem 23, then under the homomorphism ν^* of Theorem 22 the image $\nu^*(\sigma)$ is the linear functional whose value on an element $f = (f_\alpha) \in \Gamma(M, \mathcal{O}(\eta^n\kappa^{n+1}\lambda^{-n}))$ is given by

$$\nu^*(\sigma)\cdot f = 2\pi i\, \mathcal{R}(fH) .$$

Proof. If $(\sigma_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{P}_n(\lambda^{-n}))$ is a cocycle representing the cohomology class σ, then it is readily verified that the condition that σ be represented by the section $(h_\alpha) = (d^{n+1}H_\alpha) \in \Gamma(M, d^{n+1}\mathcal{M}(\lambda^{-n}))$ under the isomorphism given by Theorem 23 is just that

$$\sigma_{\alpha\beta}(z_\beta) = H_\beta(z_\beta) - \lambda_{\alpha\beta}(z_\beta)^n H_\alpha(z_\alpha) . \tag{38}$$

Suppose that the projective coordinate covering $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ is so chosen that the poles of the functions $H_\alpha(z_\alpha)$ are each contained in only a single coordinate neighborhood. Multiplying the functions $H_\alpha(z_\alpha)$ by C^∞ functions which differ from one only in small neighborhoods of these poles, and which vanish identically in some neighborhoods of these poles, yields C^∞ functions $g_\alpha(z_\alpha)$ which also satisfy equation (38) in each intersection $U_\alpha \cap U_\beta$. Recalling the explicit description of the mapping ν^*, as in (33), it follows that

$$\nu^*(\sigma)\cdot f = \iint_M \bar{\partial} g_\alpha \wedge f_\alpha dz_\alpha .$$

The latter integral will vanish identically except for those neighborhoods U_α containing poles of the functions $H_\alpha(z_\alpha)$; for outside of these neighborhoods, $g_\alpha(z_\alpha) = H_\alpha(z_\alpha)$ is holomorphic. For such a neighborhood U_α, however,

$$\begin{aligned}\iint_{U_\alpha} \bar{\partial} g_\alpha \wedge f_\alpha dz_\alpha &= \iint_{U_\alpha} \bar{\partial}(f_\alpha g_\alpha dz_\alpha) = \iint_{U_\alpha} d(f_\alpha g_\alpha dz_\alpha) \\ &= \iint_{\partial U_\alpha} f_\alpha g_\alpha dz_\alpha = \int_{\partial U_\alpha} f_\alpha H_\alpha dz_\alpha \\ &= 2\pi i \,\mathcal{R}(fH) ,\end{aligned}$$

which yields the desired result.

(e) The projective structures and their associated coordinate cohomology classes can be given a geometrically appealing global formulation as follows. Let $\tilde{M}$ be the universal covering space of a surface M, with covering mapping $\pi: \tilde{M} \longrightarrow M$, and let $\pi_1(M)$ be the fundamental group of the surface M. As is familiar, $\pi_1(M)$ can be viewed as a group of homeomorphisms of

$\tilde{M}$ onto itself, commuting with the covering mapping π and such that $\tilde{M}/\pi_1(M) = M$. (Identifying the group $\pi_1(M)$ as defined in (b) above with the usual fundamental group, this interpretation can be found for instance in Seifert-Threlfall, Lehrbuch der Topologie, chapter 8 (Chelsea, 1947). Alternatively, this result can be derived directly from the discussion in (b), parodying in simplicial terms the standard construction.) If M has a projective structure, it induces a unique projective structure on $\tilde{M}$ by means of the mapping $\pi: \tilde{M} \longrightarrow M$. For if $\{U_\alpha, z_\alpha\}$ is a projective coordinate covering of M such that the sets U_α are connected and simply-connected, then each connected component of $\pi^{-1}(U_\alpha) \subset \tilde{M}$ will be homeomorphic to U_α under the mapping π, and the functions $z_\alpha \circ \pi$ can thus be used as coordinate mappings on each such component; it is evident that this is a projective coordinate covering of $\tilde{M}$, and that equivalent projective coordinate coverings of M induce equivalent projective coordinate coverings of $\tilde{M}$. Note that the mappings in $\pi_1(M)$ are projective transformations of $\tilde{M}$ for the given projective structure, in the sense that they are represented by projective functions in terms of local coordinates for any projective coordinate covering belonging to that structure; this is quite obvious, since indeed the mappings are represented by the identity functions in the above coordinate covering.

Now since $\pi_1(\tilde{M}) = 1$, it follows from Lemma 27 that the coordinate cohomology class of any projective structure on $\tilde{M}$ is trivial; hence there is a projective coordinate covering $\{U_\alpha, z_\alpha\}$ of $\tilde{M}$ representing the given projective structure and such that the coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$ are identity mappings. The various coordinate mappings z_α then define a global mapping $\rho: \tilde{M} \longrightarrow D$ from $\tilde{M}$ onto a subset $D \subset \mathbb{P}$. The mapping ρ is a local homeomorphism, so the image D is a connected open subset of the projective line $\mathbb{P}$. Note that any other such coordinate covering representing the same projective structure will define a mapping $\rho_1: \tilde{M} \longrightarrow D_1$ also, but $\rho_1 = R\circ\rho$ for some projective transformation R; to this extent the mapping ρ is determined uniquely by the projective structure. If $\tilde{T} \in \pi_1(\tilde{M})$ is a covering translation mapping, then $\tilde{T}$ will be represented by a projective transformation in terms of any local coordinates for the given projective structure of $\tilde{M}$; thus for any point $\tilde{p}_0 \in \tilde{M}$ there will be an element $T \in PL(2,\mathbb{C})$ such that $\rho(\widetilde{Tp}) = T(\rho\tilde{p})$ for all points $\tilde{p}$ near $\tilde{p}_0$, since ρ is defined by the local coordinate mappings. Clearly the element $T \in PL(2,\mathbb{C})$ is independent of the point $\tilde{p}_0 \in \tilde{M}$; for the correspondence $\tilde{p}_0 \longrightarrow T$ is locally constant as noted above, and $\tilde{M}$ is connected. Therefore for any transformation $\tilde{T} \in \pi_1(\tilde{M})$ there is an element $T \in PL(2,\mathbb{C})$ such that

$$(39) \qquad \rho(\tilde{T}\tilde{p}) = T\rho(\tilde{p}) \quad \text{for all } \tilde{p} \in \tilde{M} .$$

It is evident that each T necessarily maps the domain D onto itself; and that the mapping $\rho^*: \pi_1(\tilde{M}) \longrightarrow PL(2,\mathbb{C})$ defined by $\rho^*(\tilde{T}) = T$, where $\tilde{T}$ and T are related by (39), is a group homomorphism. Note that if $\rho_1 = R\circ\rho$ is another such mapping representing the same projective structure, then $\rho_1(\tilde{T}\tilde{p}) = R\rho(\tilde{T}\tilde{p}) = RT\rho(\tilde{p}) = RTR^{-1}\rho_1(\tilde{p})$; hence $\rho_1^* = R\rho^*R^{-1}$. This pair of mappings

$$(40) \qquad \rho\colon \tilde{M} \longrightarrow D\ , \qquad \rho^*\colon \pi_1(\tilde{M}) \longrightarrow PL(2,\mathbb{C})\ ,$$

related by

$$(41) \qquad \rho(\tilde{T}\tilde{p}) = \rho^*(\tilde{T})\cdot\rho(\tilde{p})\ , \text{ for all } \tilde{T} \in \pi_1(\tilde{M}) \text{ and } \tilde{p} \in \tilde{M}\ ,$$

will be called a <u>geometric realization</u> of the given projective structure on M. Note that the mapping ρ is a complex analytic local homeomorphism, and that ρ^* is a group homomorphism. Two geometric realizations (ρ,ρ^*) and (ρ_1,ρ_1^*) will be called <u>equivalent</u> if there is an element $R \in PL(2,\mathbb{C})$ such that $\rho_1 = R\rho$ and $\rho_1^* = R\rho^* R^{-1}$. The previous observations show that there is a natural one-to-one correspondence between projective structures on a Riemann surface M and equivalence classes of geometric realizations; for it is apparent that any geometrical realization determines a projective structure on the surface M.

If (ρ,ρ^*) is the geometric realization of a projective structure on M, the mappings ρ^* belonging to all equivalent geometric realizations form an element $(\rho^*) \in \mathrm{Hom}(\pi_1(M),PL(2,\mathbb{C}))/PL(2,\mathbb{C})$. It is easy to see that this element is precisely the image of the coordinate cohomology class of the given projective structure under the homomorphism of Lemma 27; the verification is straightforward, and will be left to the reader. This provides the most convenient way of looking at the coordinate cohomology classes of projective structures on a Riemann surface.

As an example, consider the analogous construction for affine structures; thus, let M be a compact Riemann surface of genus 1 with a fixed affine structure, and let $\rho\colon \tilde{M} \longrightarrow D$ and

$\rho^*: \pi_1(M) \longrightarrow A$ be the geometrical realization of that affine structure, where D is a subdomain of the complex line and A is the group of affine transformations. Here of course $\pi_1(M)$ is a free abelian group on two generators $\tilde{T}_1$ and $\tilde{T}_2$; and recalling the discussion of equation (17), the homomorphism ρ^* will have one of the following forms, where $T_i = \rho^*(\tilde{T}_i)$ are the indicated affine transformations:

$$(42) \quad \begin{cases} \text{(i)} & T_1 z = z + b_1, \quad T_2 z = z + b_2 \\ \text{(ii)} & T_1^* w = a_1 w, \quad T_2^* w = a_2 w, \qquad a_1 a_2 \neq 0 . \end{cases}$$

For the quotient space $D/\rho^*\pi_1(M)$ to be compact, it is clear that in case (i) it is necessary that b_1 and b_2 be linearly independent over the reals, and $D = \mathbb{C}$; and in case (ii) it is necessary that $|a_i| \neq 1$ for either $i = 1$ or $i = 2$, and $D = \mathbb{C}^*$. In case (i) it is evident that $\rho: \tilde{M} \longrightarrow D$ is a homeomorphism and ρ^* is an isomorphism. Thus we can identify $\tilde{M} = \mathbb{C}$, $\pi_1(M) = \{$group of translations generated by T_1 and T_2 in (i)$\}$, and $M = \mathbb{C}/\pi_1(M)$. This is just the familiar representation of a compact complex torus, as discussed in §1(f), and provides the simplest affine structure on the torus. Retaining these identifications, consider the mapping $\rho: \tilde{M} \longrightarrow \mathbb{C}^*$ defined by $w = \rho(z) = e^{cz}$ for a complex constant $c \neq 0$. This is a covering mapping, exhibiting ρ as the universal covering space of $\mathbb{C}^*$. Furthermore,

$$\rho(T_i z) = \rho(z + b_i) = e^{c(z+b_i)} = a_i \rho(z) = T_i^* \rho(z) ,$$

where $a_i = e^{cb_i}$ and T_i^* are the transformations given in 42(ii). Thus evidently ρ and the homomorphism $\rho^*: T_i \longrightarrow T_i^*$ form a geometric realization of another affine structure on the same underlying Riemann surface. (Note that the affine structure 42(i) for the case $c = 0$, and these additional affine structures for all values $c \neq 0$, are easily seen to be distinct, and to be all the affine structures on that Riemann surface. Thus, as noted in part (a), the set of all affine structures on a given complex torus are in one-to-one correspondence with the complex numbers $c \in \mathbb{C}$.) These examples all have the property that $\rho: \tilde{M} \longrightarrow D$ is not just a local homeomorphism, but a covering mapping. The group $\rho^*\pi_1(M)$ does not always act in a discontinuous manner on the domain $D = \mathbf{C}^*$; so although there is always a continuous mapping $M = \tilde{M}/\pi_1(M) \longrightarrow D/\rho^*\pi_1(M)$ induced by $\rho: \tilde{M} \longrightarrow D$, it is not necessarily a homeomorphism, nor even a covering mapping. It may be observed that the same coordinate cohomology class (or equivalently the same group $\rho^*\pi_1(M)$) can be associated to affine structures on inequivalent complex tori; thus Theorem 20 definitely requires consideration of a fixed underlying complex structure.

For some further discussion of this geometric realization, see R. C. Gunning, Special coordinate coverings of Riemann surfaces, Math. Annalen, 1966.

(After this had been written there came to my attention the following paper, which contains some related results: N. S. Hawley and M. Schiffer, Half-order differentials on Riemann surfaces, Acta Math. 115 (1966), 199-236.)

§10. Representations of Riemann surfaces.

(a) Perhaps the simplest concrete representation of a Riemann surface is as a branched covering of the projective line $\mathbb{P}$. To describe the general topological situation, consider two 2-dimensional manifolds M and N. A continuous mapping $f: M \longrightarrow N$ is called a <u>local branched covering</u> if each point $p \in M$ has an open neighborhood $U \subset M$ such that $f(U)$ is open in M, and the restriction of f exhibits $U-p$ as an m-sheeted covering space of $f(U) - f(p)$ for some integer m; the integer $m-1$ will be called the <u>branching order</u> of the mapping f at the point p, and will be denoted by $o_f(p)$. The point p is called a <u>regular point</u> of f if $o_f(p) = 0$, and a <u>branch point</u> of f if $o_f(p) > 0$. The expression $\Sigma_{p \in M}\, o_f(p)$ is called the <u>total branching order</u> of the mapping f. Note that the branch points form a discrete subset of M; the mapping f is an open mapping; and f is a local homeomorphism in a neighborhood of any regular point. If M and N are Riemann surfaces, then any non-trivial complex analytic mapping $f: M \longrightarrow N$ is a local branched covering; the branching order at a point p is the order of the zero of the derivative of f in any local coordinate systems in M and N, that is, $o_f(p) = \nu_p(f')$. (For a discussion of the topological properties of analytic mappings, see for example L. Ahlfors, Complex Analysis, pp. 130-133, (McGraw-Hill, 1966).) As a converse assertion in some sense, if $f: M \longrightarrow N$ is a local branched covering between two topological manifolds and N has a complex structure, then M has a unique complex structure for which f

is an analytic mapping. The proof is straightforward, and will be left to the reader.

The global form of this situation is also of interest. A continuous mapping $f\colon M \longrightarrow N$ between two topological surfaces is called an r-sheeted branched covering if f is a local branched covering and if for every point $q \in N$,

$$\sum_{\{p \in M | f(p) = q\}} (o_f(p) + 1) = r \ .$$

Fixing $q \in N$, let $p_1, p_2, \ldots, p_s \in M$ be the points of $f^{-1}(q)$, and select open neighborhoods U_i around p_i exhibiting the local branching of f at p_i; the sets U_i can be chosen small enough to have the same image under f. The restriction of f then exhibits $\cup_i (U_i - p_i)$ as an r-sheeted covering space of $f(U_i) - q$. It is thus evident that the image under f of the branch points form a discrete subset $\{q_1, q_2, \ldots\} \subset N$; and that

$$f\colon M - \cup_i f^{-1}(q_i) \longrightarrow N - \cup_i q_i$$

is an r-sheeted covering space in the ordinary sense.

Theorem 24. Let M be a compact Riemann surface of genus g, and $\xi \in H^1(M, \mathcal{O}^*)$ be a complex line bundle of Chern class $c(\xi) = r$. Then to any pair of complex analytic sections $f_0, f_1 \in \Gamma(M, \mathcal{O}(\xi))$ which have no common zero on M there is canonically associated a complex analytic mapping

$$f = (f_0, f_1)\colon M \longrightarrow \mathbb{P} \ ,$$

which is an r-sheeted branched covering with total branching order $2(g + r - 1)$.

Proof. Let $\mathfrak{U} = \{U_\alpha, z_\alpha\}$ be a complex analytic coordinate covering of M, and $(\xi_{\alpha\beta}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ be a cocycle representing the line bundle ξ. The sections f_i are represented by complex analytic functions $f_{i\alpha}(z_\alpha)$, such that $f_{i\alpha}(z_\alpha(p)) = \xi_{\alpha\beta}(p) f_{i\beta}(z_\beta(p))$ whenever $p \in U_\alpha \cap U_\beta$. The map from U_α to $\mathbb{P}$, defined in terms of homogeneous coordinates on $\mathbb{P}$ by $p \longrightarrow (f_{0\alpha}(z_\alpha(p)), f_{1\alpha}(z_\alpha(p))) \in \mathbb{P}$, is clearly complex analytic; and the two maps thus defined in $U_\alpha \cap U_\beta$ agree, since $(f_{0\beta}, f_{1\beta})$ and $(\xi_{\alpha\beta} f_{0\beta}, \xi_{\alpha\beta} f_{1\beta}) = (f_{0\alpha}, f_{1\alpha})$ represent the same point in $\mathbb{P}$. This then defines a complex analytic mapping $f\colon M \longrightarrow \mathbb{P}$, which exhibits M as a local branched covering of $\mathbb{P}$. Next, for any points $a = (a_0, a_1) \in \mathbb{P}$ and $p \in M$, observe that $f(p) = a$ precisely when $a_1 f_0(p) - a_0 f_1(p) = 0$; thus the points $p_i \in M$ such that $f(p_i) = a$ are precisely the zeros of the complex analytic section $h = a_1 f_0 - a_0 f_1 \in \Gamma(M, \mathcal{O}(\xi))$. Furthermore, if $p_i \in U_\alpha$ and if say $a_1 \neq 0$, then in terms of inhomogeneous local coordinates in $\mathbb{P}$ the mapping f has the local description

$$p \longrightarrow \frac{f_{0\alpha}(z_\alpha(p))}{f_{1\alpha}(z_\alpha(p))} = \frac{a_0}{a_1} + \frac{h_\alpha(z_\alpha(p))}{a_1 f_{1\alpha}(z_\alpha(p))}$$

for p near p_i; consequently

$$o_f(p_i) = \nu_{p_i}(f_{0\alpha}/f_{1\alpha})' = \nu_{p_i}(h_\alpha/a_1 f_{1\alpha}) - 1 = \nu_{p_i}(h_\alpha) - 1 ,$$

since $f_{1\alpha}(p_i) \neq 0$. From Theorem 11 it then follows that

$$r = c(\xi) = \sum_i \nu_{p_i}(h) = \sum_i (o_f(p_i) + 1) ;$$

so that $f\colon M \longrightarrow \mathbb{P}$ is actually an r-sheeted branched covering.

Finally, introduce the analytic functions

$$g_\alpha(z_\alpha) = \det\begin{vmatrix} f_{0\alpha}(z_\alpha) & f'_{0\alpha}(z_\alpha) \\ f_{1\alpha}(z_\alpha) & f'_{1\alpha}(z_\alpha) \end{vmatrix}$$

in the various neighborhoods $z_\alpha(U_\alpha)$. Since in terms of inhomogeneous local coordinates the mapping f has the local description $p \longrightarrow f_{0\alpha}(z_\alpha(p))/f_{1\alpha}(z_\alpha(p))$ wherever $f_1(p) \neq 0$, it follows that $o_f(p) = \nu_p(f_{0\alpha}/f_{1\alpha})' = \nu_p(-g_\alpha f_{1\alpha}^{-2}) = \nu_p(g_\alpha)$; and the same result holds at those points where $f_0(p) \neq 0$. The total branching order is then just the total order of the functions g_α . Note that for points $p \in U_\alpha \cap U_\beta$,

$$f'_{i\alpha}(z_\alpha(p)) = \frac{d}{dz_\alpha}[\xi_{\alpha\beta}(z_\beta(p))\cdot f_{i\beta}(z_\beta(p))]$$

$$= \kappa_{\alpha\beta}[\xi_{\alpha\beta}(z_\beta(p))\cdot f'_{i\beta}(z_\beta(p)) + \xi'_{\alpha\beta}(z_\beta(p))\cdot f_{i\beta}(z_\beta(p))],$$

where $\kappa_{\alpha\beta} = \dfrac{dz_\beta}{dz_\alpha}$ define the canonical bundle of M ; it readily follows from this that $g_\alpha(z_\alpha(p)) = \kappa_{\alpha\beta}(p)\xi^2_{\alpha\beta}(p)g_\beta(z_\beta(p))$, so that $(g_\alpha) \in \Gamma(M, \mathcal{O}(\kappa\xi^2))$. Applying Theorem 11 once again, the total branching order b is

$$b = \sum_{p \in M} o_f(p) = \sum_{p \in M} \nu_p(g_\alpha) = c(\kappa\xi^2) = 2(r+g-1) ,$$

thus completing the proof.

Now suppose that f is a non-constant meromorphic function on a compact Riemann surface M ; as in §1(e) that function can be considered as an analytic mapping $f\colon M \longrightarrow \mathbb{P}$. Note that the divisor of the function f can be written in the form $\mathcal{D}(f) = \sum_{i=1}^{r} (1\cdot p_i - 1\cdot q_i)$ where $p_i \neq q_j$, for from Theorem 11 the total order of the divisor of f must be zero; the integer r, the total order of the zeros of f, will be called the degree of that function.

Corollary 1. If f is a non-constant meromorphic function of degree r on a compact Riemann surface M of genus g, then the analytic mapping $f: M \longrightarrow \mathbb{P}$ exhibits M as an r-sheeted branched covering of $\mathbb{P}$ with total branching order $2(g+r-1)$.

Proof. Writing the divisor of the function f in the form $\mathcal{J}(f) = \sum_{i=1}^{r} (1\cdot p_i - 1\cdot q_i)$ where $p_i \neq q_j$, consider the complex line bundle $\xi = \zeta_{p_1}\cdots\zeta_{p_r} = \zeta_{q_1}\cdots\zeta_{q_r}$, where ζ_p are the point bundles considered in §7(c). There are analytic sections $f_0, f_1 \in \Gamma(M, \mathcal{O}(\xi))$ such that $\mathcal{J}(f_0) = \sum_{i=1}^{r} 1\cdot p_i$ and $\mathcal{J}(f_1) = \sum_{i=1}^{r} 1\cdot q_i$; and $f = f_0/f_1$. The functions f_0 and f_1 have no common zeros, and the mapping $f: M \longrightarrow \mathbb{P}$ defined by the meromorphic function f coincides with the mapping $(f_0, f_1): M \longrightarrow \mathbb{P}$ constructed in Theorem 24. The desired result thus follows immediately from that theorem.

Corollary 2. If $\xi \in H^1(M, \mathcal{O}^*)$ is a complex line bundle on a compact Riemann surface M, such that $c(\xi) = r$ and $\gamma(\xi) \geq 2$, then to any pair of linearly independent analytic sections $f_0, f_1 \in \Gamma(M, \mathcal{O}(\xi))$ there is canonically associated a complex analytic mapping

$$f = (f_0, f_1): M \longrightarrow \mathbb{P}$$

which exhibits M as a branched covering of $\mathbb{P}$ of at most r sheets.

Proof. Let $\mathcal{J}_0 = \sum_{i=1}^{s} 1\cdot p_i$ be the divisor of the common zeros of the functions f_0 and f_1, counting multiplicities; and choose an analytic section $g \in \Gamma(M, \mathcal{O}(\eta))$ of the line bundle $\eta = \zeta_{p_1}\cdots\zeta_{p_s}$ such that $\mathcal{J}(g) = \mathcal{J}_0$. Then f_0/g and f_1/g are complex analytic sections of the complex line bundle ξ_η^{-1}, and these sections have no common zeros; so by Theorem 24 the mapping $(f_0/g, f_1/g): M \longrightarrow \mathbb{P}$ exhibits M as a branched covering of $\mathbb{P}$ having $c(\xi_\eta^{-1}) = r-s \leq r$ sheets. Note that outside of the points of the divisor $\mathcal{J}_0$ the function g is non-vanishing, so that $(f_0/g, f_1/g)$ and (f_0, f_1) define the same mapping of M to $\mathbb{P}$; the mapping is thereby canonically determined by the sections f_0 and f_1 alone.

Corollary 3. Any compact Riemann surface M can be represented as a branched covering $f: M \longrightarrow \mathbb{P}$ of the projective line; the genus g of M, the number r of sheets, and the total branching order b are related by $b = 2(g+r-1)$.

Proof. Since every compact Riemann surface admits a non-constant meromorphic function by the fundamental existence theorem, Theorem 12, this assertion follows from Corollary 1; it is merely inserted for the sake of explicitness.

Remarks. It is clear that any finitely-sheeted branched covering of the projective line $\mathbb{P}$ is a Riemann surface, with a unique complex structure for which the covering mapping is an analytic mapping. Then the genus can be calculated from the branching order and the number of sheets by applying the formula in Theorem 24. The genus can also be calculated directly in a purely topological manner, without reference to the analytic structure, as follows. Letting $f\colon M \longrightarrow \mathbb{P}$ be an r-sheeted branched covering, triangulate the surface $\mathbb{P}$ in such a manner that the images under f of the branch points are vertices of the triangulation. The triangulation can then be lifted back to a triangulation of M under the mapping f; it is only necessary to assume the triangulation of $\mathbb{P}$ fine enough that the interiors of the one- and two-simplices are homeomorphic to each component of their inverse images under f. Letting n_i be the number of i-simplices in the triangulation of $\mathbb{P}$, it is evident that the induced triangulation of M will have rn_0-b zero-simplices, rn_1 one-simplices, and rn_2 two-simplices. Thus the Euler characteristics (see Seifert-Threlfall, <u>Lehrbuch der Topologie</u>, §23, (Chelsea, 1947)) of $\mathbb{P}$ and M are related as follows:

$$\chi(M) = (rn_0-b) - (rn_1) + (rn_2) = r(n_0 - n_1 + n_2) - b = r\cdot\chi(\mathbb{P}) - b \ .$$

On the other hand, these Euler characteristics are also given by

$$\chi(\mathbb{P}) = 2 \ , \qquad \chi(M) = 2 - 2g \ ,$$

where g is the genus of M. Hence $b = 2(r+g-1)$, the desired formula.

If M is a compact Riemann surface of genus g, it follows from the Riemann-Roch theorem (recalling in particular the table in formula 14 of §7) that $\gamma(\xi) \geqq 2$ for any line bundle ξ for which $c(\xi) \geqq g+1$; hence by Corollary 2 of Theorem 24 the surface M can be represented as a branched covering of $\mathbb{P}$ of at most $g+1$ sheets. This is far from being the best possible result in general; we shall return to this question again later, but for the present merely consider some simple results relating to the Weierstrass points on the surface. For any point $p \in M$, let r be the least non-gap in the Weierstrass gap sequence at p; it then follows from Theorem 14 that $\gamma(\zeta_p^r) = 2$, where ζ_p is the point bundle associated to the point p. Thus M can be represented as a branched covering of $\mathbb{P}$ of at most r sheets. Actually, an even more precise assertion can be made. There will exist a meromorphic function f on the surface M, having as its only singularity a pole at p of order precisely r, by Theorem 14 again. From Corollary 1 of Theorem 24, the function f considered as an analytic mapping $f: M \longrightarrow \mathbb{P}$ will exhibit M as an r-sheeted branched covering of $\mathbb{P}$; the point $p \in M$ will be the only point of M covering the point at infinity on $\mathbb{P}$, and hence will be a branch point of order $r-1$ on the surface. Conversely, it is clear that whenever $f: M \longrightarrow \mathbb{P}$ is an r-sheeted branched covering such that a point $p \in M$ is a branch point of order $r-1$, then r is a non-gap in the Weierstrass gap sequence at p; for the image $f(p) \in \mathbb{P}$ can always be taken to the point at infinity on $\mathbb{P}$ by a projective transformation, and the composition of f with that projective transformation will be a meromorphic function whose sole singularity is a pole at p of order r.

For a general point $p \in M$ the first non-gap value is $r = g+1$; however when $g > 1$ there are always Weierstrass points, and at any such point the first non-gap will satisfy the inequality $2 \leq r \leq g$. At a normal Weierstrass point the value will be $r = g$, and at a hyperelliptic Weierstrass point the value will be $r = 2$.

In the latter case considerably more can be asserted, as follows.

Theorem 25. A compact Riemann surface of genus $g > 1$ is hyperelliptic if either of the following two conditions holds:

(i) the surface has a hyperelliptic Weierstrass point;

(ii) the surface has a complex line bundle ξ with $c(\xi) = 2$ and $\gamma(\xi) = 2$.

The hyperelliptic surfaces are precisely those which can be represented as two-sheeted branched covering surfaces of the projective line $\mathbb{P}$, and which are of genus $g > 1$; the branch points are precisely the Weierstrass points, all are hyperelliptic Weierstrass points, and there are $2(g+1)$ of these points.

Proof. Note first that condition (i) implies condition (ii); for if p is a hyperelliptic Weierstrass point on the Riemann surface M , then $c(\zeta_p^2) = 2$ and $\gamma(\zeta_p^2) = 2$. Now if condition (ii) holds, it follows from Corollary 2 of Theorem 24 that the surface M can be represented as a branched covering of $\mathbb{P}$ with at most 2 sheets; and since genus $g > 1$ means that $M \neq \mathbb{P}$, there will be exactly 2 sheets. Each branch point will necessarily have branching order 1 ; and since by Theorem 24 the total branching order is $b = 2(g+1)$, there will be altogether $2(g+1)$ of these branch points. At each branch point the value $r = 2$ will be a non-gap, as noted in the above discussion; hence all these branch points will be hyperelliptic Weierstrass points. Recalling Theorem 16, it follows immediately that these are all the Weierstrass branch points, and hence the surface is a hyperelliptic surface. Since this argument only used the fact that M could be represented as a two-sheeted covering of $\mathbb{P}$, all the assertions of the theorem have been proved.

If M is a compact Riemann surface of genus $g = 1$, then for any line bundle ξ with $c(\xi) = 2$ it follows from the Riemann-Roch theorem that $\gamma(\xi) = 2$; so that M can be represented as a branched two-sheeted covering of $\mathbb{P}$, having necessarily $2(g+1) = 4$ branch points. This again illustrates the similarities between elliptic and hyperelliptic Riemann surfaces. If M has genus $g = 2$, then necessarily M is hyperelliptic. To see this, recall that for genus $g = 2$ the Weierstrass gap sequence has the form $1 = \rho_1 < \rho_2 < 4$, so that either $\rho_2 = 2$ (and p is not a Weierstrass point) or $\rho_2 = 3$ (and p is a hyperelliptic Weierstrass point); since M has at least one Weierstrass point by Theorem 15, it follows immediately that M is hyperelliptic. We shall see later that not all surfaces of genus $g > 2$ are hyperelliptic; and also that the surfaces of genus g can be represented by branched coverings of fewer than g sheets, if it is not required that all the sheets meet at some point.

(b) The preceding representation of a compact Riemann surface as a branched covering of the projective line can be used to provide a useful description of the global meromorphic functions on the surface. The set $\mathfrak{M}_M$ of all meromorphic functions on any Riemann surface M is a field, under the operations of pointwise addition and multiplication of functions. The field $\mathfrak{M}_M$ contains the subfield $\mathbb{C}$ of complex constants; and for any element

$f \in \mathcal{M}_M$ the field $\mathcal{M}_M$ also contains the subfield $\mathbb{C}(f)$ of rational functions of f. (Recall that for any fields $E \subset F$ and for any element $x \in F$, the set of all polynomials in x with coefficients in E is an integral domain denoted by $E[x]$; and the smallest subfield of F containing both E and x is the set of all quotients of polynomials in x with coefficients in E, or in other words the set of all rational functions of x, and is denoted by $E(x)$. For the elementary properties of fields which will be used here, see for instance B. L. van der Waerden, Modern Algebra vol. I, (Frederick Ungar Co., N.Y., 1949).) As a simple preliminary, note the following result.

<u>Lemma 30.</u> If f is a meromorphic function of degree 1 on the projective line $\mathbb{P}$, then $\mathcal{M}_{\mathbb{P}} = \mathbb{C}(f)$.

Proof. Let z be the inhomogeneous coordinate on the projective line $\mathbb{P}$. Note that by Corollary 1 to Theorem 24, the mapping $f: \mathbb{P} \longrightarrow \mathbb{P}$ is an analytic homeomorphism. Then for any points $p \neq q$ in $\mathbb{P}$, the function $(f(z) - f(p))/(f(z) - f(q))$ is meromorphic on $\mathbb{P}$ and has divisor precisely $1\cdot p - 1\cdot q$; in case that either p or q is the pole of f, the obvious modifications of this formula will be left to the reader. If $g \in \mathcal{M}_{\mathbb{P}}$ is any meromorphic function, with divisor $\mathcal{J}(g) = \Sigma_i (p_i - q_i)$, it is clear that

$$g(z) = C\, \Pi_i \frac{f(z) - f(p_i)}{f(z) - f(q_i)}$$

for some constant C, hence that $g \in \mathbb{C}(f)$. This suffices to prove the assertion of the lemma.

It should be noted that the inhomogeneous coordinate z on $\mathbb{P}$ can be considered as a meromorphic function of degree 1 on $\mathbb{P}$; and hence by the above Lemma, $\mathcal{M}_{\mathbb{P}} = \mathbb{C}(z)$. The following generalization of this lemma is quite straightforward.

Theorem 26. Let M be a compact Riemann surface, and $f \in \mathcal{M}_M$ be a meromorphic function of degree $r > 0$ on M . Then for any function $g \in \mathcal{M}_M$ there is a polynomial $P(x,y) \in \mathbb{C}[x,y]$ in two variables, of degree at most r in the second variable y , such that $P(f,g) \equiv 0$. (Note that $P(f,g)$ is a well-defined meromorphic function on the Riemann surface M.)

Proof. By Corollary 1 to Theorem 24, the function f considered as a mapping $f: M \longrightarrow \mathbb{P}$ exhibits M as an r-sheeted branched covering of the projective line $\mathbb{P}$. The image under f of the branch points then forms a finite set of points $q_1,\ldots,q_s \in \mathbb{P}$, and the mapping

$$f: M - \cup_i f^{-1}(q_i) \longrightarrow \mathbb{P} - \cup_i q_i$$

is an r-sheeted covering space in the ordinary sense. For each point $q \in \mathbb{P}$ which is not one of the branch points $q_1,\ldots,q_s$, select a contractible open coordinate neighborhood V of q in $\mathbb{P}$ which is regularly covered under the mapping f . Thus $f^{-1}(V) = \bigcup_{i=1}^{r} U_i$, where the U_i are disjoint open subsets of M homeomorphic to V ; let $\varphi_i: V \longrightarrow U_i$ be the analytic homeomorphisms such that $\varphi_i \circ f: U_i \longrightarrow U_i$ is the identity for each i. For any meromorphic function g on M , introduce the function

$$(1) \qquad F_V(z,y) = \prod_{i=1}^{r} (y - g\circ\varphi_i(z)) ;$$

this function is a polynomial in y of degree r, and its coefficients are meromorphic in z for $z \in V$. By construction of course, when $p \in f^{-1}(V)$,

$$F_V(f(p),g(p)) = \Pi_i \; (g(p) - g\circ\varphi_i\circ f(p)) = 0 ,$$

since $g\circ\varphi_i\circ f(p) = g(p)$ if $p \in U_i$. The same construction can be carried out in any other such coordinate neighborhood W, yielding another function $F_W(z,y)$ of a similar form. In an intersection $V \cap W$ the mappings $\varphi_{i,V}$ and $\varphi_{j,W}$ coincide in some order; the coefficients of the polynomial (1) are the elementary symmetric functions of the values $g\circ\varphi_i(z)$, and hence are independent of the ordering. Therefore $F_V(z,y) = F_W(z,y)$ for $z \in V \cap W$; and hence there is a well defined function $F(z,y)$, a polynomial in y of degree r with coefficients which are meromorphic functions on $\mathbb{P} - \bigcup_i q_i$, such that $F(f,g) \equiv 0$. If the function g is analytic at the points $f^{-1}(q_i)$, it is clear that the coefficients of the polynomial $F(z,y)$ are bounded analytic functions of z in a punctured disc centered at q_i; hence by Riemann's removable singularity theorem, the coefficients remain analytic at the point q_i. It is a straightforward matter, which will be left to the reader to verify, that the coefficients of $F(z,y)$ are meromorphic at those points q_i such that $f^{-1}(q_i)$ contains poles of g. Thus these coefficients are meromorphic on all of $\mathbb{P}$, hence are rational functions of the inhomogeneous coordinate z of $\mathbb{P}$, by Lemma 30; multiplying $F(z,y)$ by a suitable polynomial in z will therefore yield a polynomial $P(z,y)$ in two variables, with all the desired properties.

Corollary. The field of meromorphic functions on a compact Riemann surface is an algebraic function field in one variable over the complex numbers, that is, is a finite algebraic extension of a simple transcendental extension of the field $\mathbb{C}$.

Proof. If $f \in \mathcal{M}_M$ is any non-constant meromorphic function on the Riemann surface M, the field $\mathbb{C}(f)$ is a simple transcendental extension of the field $\mathbb{C}$; for otherwise f would be the root of a polynomial with coefficients in $\mathbb{C}$, hence would necessarily be a constant. If $\mathbb{C}(f)$ is not the full field $\mathcal{M}_M$, select any meromorphic function $g_1 \in \mathcal{M}_M - \mathbb{C}(f)$ and consider the field $E_1 = \mathbb{C}(f, g_1)$; by Theorem 26, E_1 is an algebraic extension of $\mathbb{C}(f)$ of degree at most r, where r is the degree of the meromorphic function f. If E_1 is not the full field $\mathcal{M}_M$, select another function $g_2 \in \mathcal{M}_M - E_1$ and consider the field $E_2 = \mathbb{C}(f, g_1, g_2)$; since g_2 is algebraic over $\mathbb{C}(f)$ of degree at most r, the field E_2 is an algebraic extension of E_1 of degree at most r as well. This process can of course be continued. However, by the theorem of the primitive element (cf. van der Waerden, page 126), the extension $E_m = \mathbb{C}(f, g_1, \ldots, g_m)$ of $\mathbb{C}(f)$ can be generated by a single element $c_1g_1 + c_2g_2 + \cdots + c_mg_m$, for some complex constants $c_1, \ldots, c_m$; so that actually E_m must have degree at most r over $\mathbb{C}(f)$ for all m. The process then necessarily stops after finitely many stages, and the result is thereby demonstrated.

Now on a compact Riemann surface M select any two meromorphic functions f, g which generate the function field of the surface, that is, which are such that $\mathfrak{M}_M = \mathbb{C}(f,g)$; and let $P(x,y)$ be the polynomial such that $P(f,g) \equiv 0$, noting that $P(x,y)$ can always be taken to be an irreducible polynomial. It is clear that this polynomial completely describes the function field of the Riemann surface, as an abstract field. It is indeed even true that the polynomial describes the Riemann surface itself, in a sense. It is more convenient for this purpose to pass from the polynomial $P(x,y)$ to a naturally associated homogeneous polynomial. Formally, write $x = t_1/t_0$ and $y = t_2/t_0$. Then if n is the degree of the polynomial $P(x,y)$, consider the homogeneous polynomial of degree n in three variables defined by

$$P_0(t_0,t_1,t_2) = t_0^n P(t_1/t_0, t_2/t_0) .$$

This will be called the <u>homogeneous form</u> of the polynomial $P(x,y)$; it is canonically determined by $P(x,y)$, and the original polynomial can be recovered by noting that $P(x,y) = P_0(1,x,y)$. Considering (t_0,t_1,t_2) as homogeneous coordinates in two-dimensional complex projective space $\mathbb{P}^2$, although $P_0(t_0,t_1,t_2)$ is not a well-defined function on $\mathbb{P}^2$, its zero locus is a well-defined subset of $\mathbb{P}^2$; for if $P_0(t_0,t_1,t_2) = 0$, then from homogeneity it follows that $P_0(tt_0,tt_1,tt_2) = t^n P_0(t_0,t_1,t_2) = 0$. The subset

$$\text{loc } P_0 = \{(t_0,t_1,t_2) \in \mathbb{P}^2 \mid P_0(t_0,t_1,t_2) = 0\}$$

is called an <u>algebraic plane curve</u> of degree n , defined by the polynomial P_0 . (It is assumed that the reader is acquainted with the elementary properties of projective spaces; see for instance W. V. D. Hodge and D. Pedoe, Methods of Algebraic Geometry, volume I, chapter V, (Cambridge University Press, 1953), for a treatment of this topic.) In the coordinate neighborhood $t_0 \neq 0$ in $\mathbb{P}^2$, with local coordinate $x = t_1/t_0$ and $y = t_2/t_0$, the curve is given by the equation $P_0(1,x,y) = P(x,y) = 0$; so the algebraic plane curve loc P_0 in projective space is in the natural sense just the extension of the curve $P(x,y) = 0$ defined in the ordinary space $\mathbb{C}^2$ of the two complex variables x,y. Note that the intersection of the curve loc P_0 with the line at infinity $t_0 = 0$ in $\mathbb{P}^2$ consists of a finite number of points; these are the points with homogeneous coordinates satisfying $P_0(0,t_1,t_2) = 0$, or in terms of the inhomogeneous coordinate $t = t_2/t_1$ on the line at infinity, the points $P_0(0,1,t) = 0$. Removing these finitely many points from loc P_0 yields the curve $P(x,y) = 0$. In the relative topology as a subset of $\mathbb{P}^2$, the set loc P_0 is just the compactification of the curve $P(x,y) = 0$, obtained by adding a finite number of points to that curve.

Lemma 31. To any algebraic plane curve there is canonically associated a compact Riemann surface.

Proof. Consider first that portion of the curve loc P_0 in the coordinate neighborhood $t_0 \neq 0$; this is just the point set loc $P = \{(x,y) \in \mathbb{C}^2 | P(x,y) = 0\}$. View $P(x,y)$ as a polynomial in y with coefficients in $\mathbb{C}(x)$, assuming first that it is irreducible, and let $D(x)$ be its discriminant; note that $D(x)$ is a polynomial in x, and let $\Delta = \{x_1,x_2,\ldots\}$ be the finite number of zeros of that polynomial $D(x)$. For each value

$x_0 \notin \Delta$ the polynomial $P(x,y)$ will have r distinct roots, where r is the degree of that polynomial in the variable y ; and letting these values be $y_1,\ldots,y_r$, since they are simple roots it follows that $\partial P(x_0,y_i)/\partial y \neq 0$. By the implicit function theorem, there are r complex analytic functions $\varphi_1(x)$, $i = 1,\ldots,r$, defined in an open neighborhood of x_0 and such that

$$\varphi_i(x_0) = y_i \quad \text{and} \quad P(x,\varphi_i(x)) \equiv 0 .$$

The points $(x,\varphi_i(x))$ are thus all the points for which $P(x,y) = 0$. It is then clear that under the natural projection $\pi\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^1$ defined by $\pi(x,y) = x$, the curve $\{(x,y) \in \mathbb{C}^2 | P(x,y) = 0,\ x \notin \Delta\}$ is an r-sheeted covering space of the set $\mathbb{C} - \Delta = \{x \in \mathbb{C} | D(x) \neq 0\}$. This provides that portion of the curve with a Riemann surface structure. Then for a point $x_1 \in \Delta$, the curve $P(x,y) = 0$ will locally provide an r-sheeted covering of a punctured disc centered at the point x_1 . These coverings are all well known, however; each connected component can be completed as a branched covering of the full disc centered at x_1 , and the Riemann surface structure extends uniquely to this completion. This associates a Riemann surface to that portion $\text{loc}\ P \subset \text{loc}\ P_0$ of the curve in the given coordinate neighborhood. Note that, except possibly at these branch points, the underlying point set of the associated Riemann surface can be identified with the point set $\text{loc}\ P$; and the identification is a topological homeomorphism. Moreover, again excepting the finitely many branch points, it is evident from the local parametrizations $y = \varphi_i(x)$ that the germs of analytic functions on $\text{loc}\ P$ in terms of its structure as a

Riemann surface are precisely the restrictions to loc P of germs of analytic functions of two complex variables defined in a neighborhood in the projective plane $\mathbb{P}^2$. Recalling from Lemma 2 that a Riemann surface structure is determined completely by the sheaf of germs of holomorphic functions on the surface, and observing from Riemann's removable singularities theorem that a mapping between two Riemann surfaces which preserves the sheaves of analytic functions except for a discrete point set is necessarily an analytic equivalence, it follows that the Riemann surface structure is uniquely determined, independently of the choices made in the above construction. Now a similar construction can be carried out in the other coordinate neighborhoods of $\mathbb{P}^2$; by the above uniqueness observation, these complex structures necessarily match in the intersections of the coordinate neighborhoods, and hence define a unique compact Riemann surface associated to the algebraic curve loc P_0 . In case the polynomial $P(x,y)$ is reducible, each irreducible factor separately determines a compact Riemann surface.

To illustrate the manner in which the Riemann surface associated to loc P_0 differs from the point set loc P_0 , consider the trivial case of the polynomial $P(x,y) = xy$. The Riemann surface corresponding to that curve consists of two disjoint copies of the complex line $\mathbb{C}$; they are imbedded in $\mathbb{C}^2$ in such a way that they intersect, but that is not reflected in the Riemann surface structure. The analytic functions on the Riemann surface are not the restrictions to the curve loc P of analytic functions of two complex variables at the origin; for the restrictions would necessarily have the same value at the intersection of the two lines, while the functions on the Riemann surface do not. For the case of an algebraic curve with singularities, such as the curve defined by the equation $P(x,y) = y^2 - x^3$ which has a singularity at the origin, the Riemann surface and the curve agree as point sets; but still the germs of analytic functions on the Riemann surface do not consist of the restrictions to the curve of germs of analytic functions of two variables at the origin. This leads further into the properties of singularities of algebraic curves than time permits for adequate treatment; the reader is referred to the standard literature on algebraic curves for a more extensive discussion.

Theorem 27. Let M be a compact Riemann surface, and $f,g \in \mathfrak{M}_M$ be meromorphic functions generating the function field of M; and let $P(x,y)$ be the irreducible polynomial such that $P(f,g) \equiv 0$. There is then a canonical analytic homeomorphism from the Riemann surface M onto the Riemann surface associated to the algebraic plane curve defined by the polynomial $P(x,y)$.

Proof. Let $\xi \in H^1(M, \mathcal{O}^*)$ be a complex line bundle on the Riemann surface M with a non-trivial holomorphic cross-section $f_0 \in \Gamma(M, \mathcal{O}(\xi))$ such that the functions $f_1 = f_0 f$ and $f_2 = f_0 g$ are holomorphic; then of course $f_i \in \Gamma(M, \mathcal{O}(\xi))$, $i = 0,1,2$. The line bundle ξ and section f_0 can be so chosen that the three functions f_0, f_1, f_2 have no common zeros on M; it suffices to select the section f_0 such that its divisor is the least for which $f_0 f$ and $f_0 g$ are holomorphic. Letting $P_0(t_0,t_1,t_2)$ be the homogeneous form of the polynomial $P(x,y)$, note that $P_0(f_0,f_1,f_2) = f_0^n P(f_1/f_0,\ f_2/f_0) = f_0^n P(f,g) \equiv 0$. Thus the mapping from M into $\mathbb{P}^2$ defined by

$$p \in M \longrightarrow (f_0(p), f_1(p), f_2(p)) \in \mathbb{P}^2$$

takes the Riemann surface M to the point set $\text{loc}\ P_0 \subset \mathbb{P}^2$ of the algebraic plane curve defined by the polynomial P. It is readily verified that this defines a complex analytic mapping from M to all points in the Riemann surface associated to the curve, except the branch points; and applying Riemann's removable singularities theorem yields an extension to all of M. The details will be left to the reader. Thus there is defined a local branched covering from M to the Riemann surface associated to the curve $P(x,y)$; and since both are compact, it follows easily that this is an r-sheeted branched covering for some index r. The meromorphic functions on M separate points, in the sense that if $p,q \in M$ and $p \neq q$ there is a meromorphic function $h \in {}_M$ such that $h(p) \neq h(q)$; for there is always a meromorphic function with a pole only at the point p. Since f,g generate the meromorphic functions, then these two functions generally separate points as well; thus the index $r = 1$, and the mapping from M to the Riemann surface associated to the curve $P(x,y)$ is one-to-one, thereby completing the proof.

<u>Corollary.</u> Two Riemann surfaces are analytically equivalent if and only if their fields of meromorphic functions are isomorphic as abstract fields.

Proof. If M, M' are two Riemann surfaces with isomorphic function fields, generators of these fields can be chosen such that they satisfy the same irreducible polynomial equation; the desired result follows immediately from Theorem 27 then.

The preceding Corollary shows that the investigation of compact Riemann surfaces can be reduced to the investigation of algebraic function fields in one variable over the complex numbers, or of algebraic plane curves. The equivalence concept for function fields is just isomorphism as abstract fields, but is slightly more complicated for algebraic curves. If $P(x,y)$ and $P'(x',y')$ define algebraic curves, then these should be considered as equivalent when their underlying function fields are the same. This means that x', y' (considered as meromorphic functions on the curve) must be rational functions of x, y, and conversely; this equivalence concept is known as birational equivalence. The algebro-geometric form of the study of compact Riemann surfaces can be phrased as the problem of studying birationally invariant properties of algebraic plane curves. This was the original form in which the subject was studied, and the reader is referred to the standard works on algebraic geometry for further reading. The principal interest in these lectures has been the analytic aspects of the subject, so the algebraic line will be pursued no further.

(c) Let M be a compact Riemann surface of genus $g > 1$, and let $h_1, h_2, \ldots, h_g \in \Gamma(M, \mathcal{O}(\kappa))$ be a basis for the space of Abelian differentials. It was noted earlier (page 119) that these sections h_i have no common zeros on the surface. Thus if z_α is a local coordinate mapping in an open set $U_\alpha \subset M$, and if $h_{i\alpha}(z_\alpha)$ are the analytic representations of the sections h_i in that coordinate neighborhood, then the values $(h_{1\alpha}(z_\alpha), \ldots, h_{g\alpha}(z_\alpha))$ can be viewed as the homogeneous coordinates of points in the projective space $\mathbb{P}^{g-1}$ of dimension $g-1$; this leads to a complex analytic mapping $H_\alpha: U_\alpha \longrightarrow \mathbb{P}^{g-1}$. Note that in $U_\alpha \cap U_\beta$ these mappings are related by $H_\alpha(z_\alpha) = = (h_{1\alpha}(z_\alpha), \ldots, h_{g\alpha}(z_\alpha)) = (\kappa_{\alpha\beta}(z_\beta)h_{1\beta}(z_\beta), \ldots, \kappa_{\alpha\beta}(z_\beta)h_{g\beta}(z_\beta)) = = H_\beta(z_\beta)$; this therefore yields a global analytic mapping $H: M \longrightarrow \mathbb{P}^{g-1}$. The mapping H is called the principal mapping, and the image $H(M) \subset \mathbb{P}^{g-1}$ is called the principal curve associated to the Riemann surface M. Note that choosing a different basis for $\Gamma(M, \mathcal{O}(\kappa))$ has the effect of replacing the principal mapping H by a non-singular linear transform of H, or equivalently, of following the mapping H by a non-singular projective transformation in $\mathbb{P}^{g-1}$. Thus the principal mapping and the principal curve are determined uniquely up to a non-singular projective transformation in $\mathbb{P}^{g-1}$.

Theorem 28 (a). If M is a compact Riemann surface of genus $g > 1$ and M is not hyperelliptic, then the principal mapping $H: M \longrightarrow \mathbb{P}^{g-1}$ is a one-to-one non-singular complex analytic mapping, and the principal curve $H(M)$ is a one-dimensional complex analytic submanifold of $\mathbb{P}^{g-1}$.

Proof. That the mapping $H: M \longrightarrow \mathbb{P}^{g-1}$ is a complex analytic mapping is obvious from the definitions. If H were not one-to-one, there would be two distinct points $p, q \in M$ such that $H(p) = H(q)$; and by a non-singular projective transformation in $\mathbb{P}^{g-1}$ that image can be taken to be the point $H(p) = H(q) = (1,0,\ldots,0) \in \mathbb{P}^{g-1}$. Thus $h_i(p) = h_i(q) = 0$ for $i = 2,\ldots,g$. Let $\xi = \zeta_p \zeta_q$, and let $f \in \Gamma(M, \mathcal{O}(\xi))$ be the standard non-trivial section, with divisor $\mathcal{J}(f) = 1 \cdot p + 1 \cdot q$. The functions h_i/f for $i = 2,\ldots,g$, then linearly independent complex analytic sections of the complex line bundle $\kappa\xi^{-1}$, so that $\gamma(\kappa\xi^{-1}) \geqq g-1$. By the Riemann-Roch Theorem, $\gamma(\xi) = \gamma(\kappa\xi^{-1}) + c(\xi) + 1-g \geqq 2$; but since $c(\xi) = 2$, it would follow from Theorem 25 that M is hyperelliptic, a contradiction. Therefore the mapping H is one-to-one. The condition that the mapping H be non-singular is just that at each point of M at least one of the coordinate functions of the mapping H be non-singular, that is, have a non-vanishing derivative. If $p \in U_\alpha$ and z_α is a local coordinate in U_α , the mapping H is given in homogeneous coordinates by $z_\alpha \longrightarrow (h_{1\alpha}(z_\alpha),\ldots,h_{g\alpha}(z_\alpha))$; and if say $h_{1\alpha}(p) \neq 0$, then in terms of the standard inhomogeneous coordinates around $H(p) \in \mathbb{P}^{g-1}$, the mapping H is described by $z_\alpha \longrightarrow (h_{2\alpha}(z_\alpha)/h_{1\alpha}(z_\alpha),\ldots,h_{g\alpha}(z_\alpha)/h_{1\alpha}(z_\alpha))$. If the mapping H were singular at p , then necessarily

$$\frac{h_{1\alpha}(p)h'_{i\alpha}(p) - h'_{1\alpha}(p)h_{i\alpha}(p)}{h_{1\alpha}(p)^2} = 0 \quad \text{for } i = 2,\ldots,g .$$

Since $h_{1\alpha}(p) \neq 0$, then writing $\underline{h}_\alpha(z_\alpha) = (h_{i\alpha}(z_\alpha))$ and

$\underline{h}'_\alpha(z_\alpha) = (h'_{i\alpha}(z_\alpha))$ as on page 117, it follows that the matrix $(\underline{h}_\alpha(p), \underline{h}'_\alpha(p))$ has rank $\rho = 1$. Then from Lemma 17 it further follows that $\gamma(\zeta_p^2) = 2$; but since $c(\zeta_p^2) = 2$, Theorem 25 implies that M is again hyperelliptic, a contradiction. The mapping H is therefore non-singular, and as an immediate consequence of that, the image curve $H(M) \subset \mathbb{P}^{g-1}$ is a one-dimensional complex analytic submanifold of the projective space, concluding the proof.

A few simple properties of the principal curve of a non-hyperelliptic Riemann surface are as follows. First, the principal curve does not lie in any proper linear subvariety of $\mathbb{P}^{g-1}$. For letting $(t_1,\ldots,t_g)$ be homogeneous coordinates in $\mathbb{P}^{g-1}$, if the principal curve $H(M)$ were contained in the linear subvariety $\Sigma_i a_i t_i = 0$, then necessarily $\Sigma_i a_i h_i(p) = 0$ for all $p \in M$; but this is impossible if not all the constants a_i vanish, since the sections $h_i \in \Gamma(M, \mathcal{O}(\kappa))$ are linearly independent. This means that a linear subvariety meets the curve $H(M)$ in a finite number of points only; indeed, the intersection consists of precisely $2g-2$ points, counting multiplicity. For letting $\varphi(t) = \Sigma_i a_i t_i$, as a complex analytic section of a complex line bundle on $\mathbb{P}^{g-1}$, the restriction of $\varphi(t)$ to $H(M)$ has a well-defined total number of zeros, which is the desired intersection multiplicity; but that restriction is the element $\varphi(H(p)) = \Sigma_i a_i h_i(p) \in \Gamma(M, \mathcal{O}(\kappa))$, and since $c(\kappa) = 2g-2$, it follows from Theorem 11 that $\varphi(H(p))$ has altogether $2g-2$ zeros, counting multiplicities. It is known that any complex analytic subvariety of $\mathbb{P}^{g-1}$ is an algebraic variety; this is Chow's theorem, (and a proof can be found for instance in R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables (Prentice-Hall, 1965), page 170). The principal curve is therefore a non-singular algebraic curve in the projective space $\mathbb{P}^{g-1}$; since the degree of a curve is its intersection multiplicity with a general linear hyperplane, it follows from the above remarks that $H(M)$ is of degree $2g-2$.

In the case of a hyperelliptic Riemann surface M, the situation is quite different. To any mapping $f: M \longrightarrow \mathbb{P}$ which exhibits M as a two-sheeted branched covering of $\mathbb{P}$ as in Theorem 25 there is canonically associated an analytic automorphism $\theta: M \longrightarrow M$ of period two. Omitting the branch points, $f: M \longrightarrow \mathbb{P}$ is an ordinary two-sheeted covering mapping, and θ is just the operation of interchanging the sheets. The operation extends analytically to the branch points in an obvious manner; for a local coordinate z can be chosen in an open neighborhood of the branch point $z = 0$ in M such that the mapping f has the form $f(z) = z^2$, and in that neighborhood $\theta z = -z$. Such a mapping θ will be called a <u>hyperelliptic automorphism.</u> As noted, θ^2 is the identity mapping. The fixed points of θ are precisely the $2g+2$ Weierstrass points on the surface M, since by Theorem 25 these are precisely the branch points of the mapping $f: M \longrightarrow \mathbb{P}$. The mapping $f: M \longrightarrow \mathbb{P}$ can be described as the quotient mapping $M \longrightarrow M/(\theta)$; for $\{1, \theta\}$ is a group of analytic automorphisms on M, and the mapping f is equivalent to collapsing M into the quotient space under this group of automorphisms. It is thus obvious that a compact Riemann surface M is hyperelliptic precisely when it admits a complex analytic automorphism $\theta\colon M \longrightarrow M$ of period two such that the quotient space $M/(\theta) \cong \mathbb{P}$; and although the fixed points of θ are unique, it is not yet evident that there is a unique such mapping θ on any hyperelliptic Riemann surface.

Theorem 28 (b). If M is a hyperelliptic Riemann surface, the principal mapping $H\colon M \longrightarrow \mathbb{P}^{g-1}$ can be factored as the composition of the quotient mapping $M \longrightarrow M/(\theta)$ modulo a hyperelliptic automorphism, and the canonical non-singular complex analytic mapping $\mathbb{P} \longrightarrow \mathbb{P}^{g-1}$ defined by

$$(2) \qquad (z_0, z_1) \longrightarrow (z_0^{g-1}, z_0^{g-2} z_1, \ldots, z_0 z_1^{g-2}, z_1^{g-1})$$

in terms of the standard homogeneous coordinates.

Proof. Let $f\colon M \longrightarrow \mathbb{P}$ be a two-sheeted branched covering such that the point with inhomogeneous coordinate ∞ on $\mathbb{P}$ is not the image of a branch point. Then considering f as a meromorphic function on M, it follows that $\mathcal{J}(f) = 1\cdot r_1 + 1\cdot r_2 - 1\cdot q_1 - 1\cdot q_2$ for some points $r_i, q_i \in M$. Considering $df \in \Gamma(M, \mathcal{M}^{1,0}) \cong \Gamma(M, \mathcal{M}(\kappa))$ as a meromorphic differential form on M, the zeros of df occur precisely at the branch points of the mapping f, and df has double poles at the points $q_i \in M$; therefore $\mathcal{J}(df) = 1\cdot p_1 + \ldots + 1\cdot p_{2g+2} - 2\cdot q_1 - 2\cdot q_2$, where $p_i \in M$ are the branch points of f, hence the Weierstrass points on M. Select a meromorphic function g on M such that $\mathcal{J}(g) = 1\cdot p_1 + \ldots + 1\cdot p_{2g+2} - (g+1)\cdot q_1 - (g+1)\cdot q_2$. (To show the existence of a function of this sort, let $q_i = f(p_i) \in \mathbb{P}$, and consider the two-valued meromorphic function y on $\mathbb{P}$ defined by $y^2 = (z-a_1) \ldots (z-a_{2g+2})$ in terms of the inhomogeneous coordinate z on $\mathbb{P}$. The function y is two-valued and has branch points at the points a_i ; hence y can be viewed as a single-valued meromorphic function on the two-sheeted covering surface M, and as such will be the desired function.) It now follows readily that for $i = 1,\ldots,g$ the differential forms $\varphi_i = f^{i-1}df/g$ are holomorphic on the Riemann surface M ; for $\mathcal{J}(\varphi_i) = (i-1)\ \mathcal{J}(f) + \mathcal{J}(df) - \mathcal{J}(g) = (i-1)\cdot r_1 + (i-1)\cdot r_2 + (g-1)\cdot q_1 + (g-1)\cdot q_2$. Since these differential forms are evidently linearly independent, they determine a basis for $\Gamma(M, \mathcal{O}^{1,0})$. In terms of this basis the principal mapping clearly has the form

$$H(p) = (1,f(p),f^2(p),\ldots,f^{g-1}(p)) . \tag{3}$$

This shows first of all that the mapping H can be factored through the mapping f ; and rewriting (3) in homogeneous form to avoid considering meromorphic functions, it is evident that H is the composition of f and the mapping (2). It is a straightforward matter, verification of which will be left to the reader, to see that (2) is a one-to-one non-singular complex analytic mapping of $\mathbb{P}$ onto $\mathbb{P}^{g-1}$, and the proof is thereby concluded.

Corollary 1. On a hyperelliptic Riemann surface M the hyperelliptic automorphism $\theta\colon M \longrightarrow M$ is unique.

Proof. The principal mapping is unique, up to composition by a non-singular projective transformation in $\mathbb{P}^{g-1}$; and since θ is determined by the principal mapping, as in Theorem 28 (b), the Corollary follows immediately.

Corollary 2. Let M be a hyperelliptic Riemann surface of genus g, with the hyperelliptic automorphism $\theta\colon M \longrightarrow M$; and let $p_1,\dots,p_{g-1}$ be any points on the surface M. Then the canonical bundle of M is given by

$$\kappa = \zeta_{p_1}\zeta_{\theta p_1}\cdots\zeta_{p_{g-1}}\zeta_{\theta p_{g-1}}$$

Proof. Let $f\colon M \longrightarrow \mathbb{P}$ be the standard two-sheeted branched covering, and let $p, \theta p \in M$ be the points which map into the point $\infty \in \mathbb{P}$ in the standard inhomogeneous coordinate covering; so that p and θp are the poles of f, considering f as a meromorphic function on M. The differential form df/g constructed in the proof of Theorem 28 (b) is holomorphic on M and has the divisor $\mathcal{D}(df/g) = (g-1)\cdot p + (g-1)\cdot\theta p$; and therefore $\kappa = \zeta_p^{g-1}\zeta_{\theta p}^{g-1}$. This is the particular case of the Corollary in which $p_1 = \dots = p_{g-1} = p$. In general, let $a_i = f(p_i)$, $a = f(p)$, be the images of these points in $\mathbb{P}$. There is a meromorphic function g^* on $\mathbb{P}$ with divisor $\mathcal{D}(g^*) = (g-1)\cdot a - \Sigma_i 1\cdot a_i$; and lifting g^* to a meromorphic function g on the Riemann surface M, by means of the mapping $f\colon M \longrightarrow \mathbb{P}$, it is clear that g will have the divisor $\mathcal{D}(g) = (g-1)\cdot(p+\theta p) - \Sigma_i 1\cdot(p_i + \theta p_i)$. It then follows that

$$\zeta_p^{g-1}\zeta_{\theta p}^{g-1} = \zeta_{p_1}\zeta_{\theta p_i}\cdots\zeta_{p_g}\zeta_{\theta p_g},$$

and hence the Corollary follows from the special case just proved.

A representation of Riemann surfaces which is closely associated to the principal curve is the following. Let $\varphi_1,\ldots,\varphi_g \in \Gamma(M, \mathcal{O}^{1,0})$. be a basis for the space of Abelian differential forms on the surface; and let $p_o \in M$ be an arbitrary but fixed base point on the surface. Choosing a basis for $H^1(M,\mathbb{Z})$, the cohomology classes $\delta\varphi_i \in H^1(M,\mathbb{C})$ are represented by vectors $(\omega_{ji}) \in \mathbb{C}^{2g}$ as on page 144; and the $2g \times g$ matrix $\Omega = (\omega_{ji})$ is the associated period matrix of the Abelian differentials on M. Recall that the Jacobi variety $\underset{\sim}{J}(M)$ of the Riemann surface M is the compact complex torus $\underset{\sim}{J}(M) = \mathbb{C}^g/{}^t\Omega\mathbb{Z}^{2g}$. Now for any point $p \in M$, select any path λ from p_o to p in M; and introduce the element

$$(4) \qquad \Phi(p) = (\textstyle\int_\lambda\varphi_1,\ldots,\int_\lambda\varphi_g) \in \mathbb{C}^g/{}^t\Omega\mathbb{Z}^{2g} = \underset{\sim}{J}(M) .$$

If λ_1 is any other path from p_o to p, then $\lambda_1 = \lambda+\tau$ where τ is a closed loop from p_o to p_o in M; and in terms of the dual basis for $H_1(M,\mathbb{Z})$, the homology class of τ will be represented by a vector $(\tau_j) \in \mathbb{Z}^{2g}$. Thus $(\int_{\lambda_1}\varphi_i) = (\int_\lambda\varphi_i + \int_\tau\varphi_i) = (\int_\lambda\varphi_i + \sum_{j=1}^{2g} \tau_j\omega_{ji}) = (\int_\lambda\varphi_i)$ in $\underset{\sim}{J}(M)$, since $\sum_{j=1}^{2g} \tau_j\omega_{ji} \in {}^t\Omega\mathbb{Z}^{2g}$. The mapping $\Phi\colon M \longrightarrow \underset{\sim}{J}(M)$ is thus well-defined, being independent of the choice of the path λ. This mapping is called a Jacobian mapping of the Riemann surface M. Note that the mapping is independent of the choice of bases for $\Gamma(M, \mathcal{O}^{1,0})$ and $H^1(M,\mathbb{C})$, in the obvious sense; but Φ does depend on the choice of the base point $p_o \in M$, a change in the base point corresponding to a translation in the Abelian group $\underset{\sim}{J}(M)$.

Theorem 29. If M is a compact Riemann surface of genus $g > 0$, then the Jacobian mapping $\Phi: M \longrightarrow \underset{\sim}{J}(M)$ is a one-to-one nonsingular complex analytic mapping, and the image $\Phi(M) \subset \underset{\sim}{J}(M)$ is a one-dimensional complex analytic submanifold of $\underset{\sim}{J}(M)$.

Proof. That the mapping Φ is a complex analytic mapping is obvious, since the integrals in (4) are complex analytic functions of the limits of integration, at least locally. If the mapping Φ were not one-to-one, there would exist distinct points $q_1, q_2 \in M$ such that $\Phi(q_1) = \Phi(q_2)$; and in terms of the explicit form (4) for that mapping, it would follow that for an arc τ from q_1 to q_2,

$$(\int_\tau \varphi_1, \ldots, \int_\tau \varphi_g) = 0 \in \mathbb{C}^g / {}^t\Omega \mathbb{Z}^{2g} = \underset{\sim}{J}(M) .$$

Then from Abel's Theorem (Theorem 18) it would further follow that $\zeta_p \zeta_q^{-1} = 1$, a contradiction (recalling the discussion on page 115). If z_α is a local coordinate mapping in a coordinate neighborhood $U_\alpha \subset M$, then writing $\varphi_i = h_{i\alpha}(z_\alpha) dz_\alpha$, the condition that the mapping Φ be singular at z_α is clearly just that $h_{1\alpha}(z_\alpha) = \ldots = h_{g\alpha}(z_\alpha) = 0$; but this can never happen, as noted on page 119. Therefore the mapping Φ is nonsingular, and therefore the image $\Phi(M)$ is a one-dimensional complex analytic submanifold of $\underset{\sim}{J}(M)$, completing the proof.

Corollary. If M is a compact Riemann surface of genus $g = 1$, then the Jacobian mapping $\Phi: M \longrightarrow \underset{\sim}{J}(M)$ is an analytic isomorphism of Riemann surfaces.

Proof. Since $\dim \underset{\sim}{J}(M) = g = 1$ in this case, the Corollary is an immediate consequence of the preceding Theorem. This provides a useful standard form for compact Riemann surfaces of genus 1 .

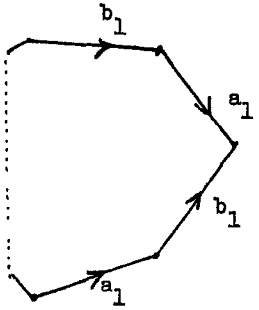
The next stage of the discussion of Riemann surfaces would involve a more detailed investigation of these last mappings (the principal and the Jacobian mappings), leading towards Torelli's Theorem and the problem of moduli of Riemann surfaces. Time has run out, however, and this must be postponed to another time.

Appendix: the topology of surfaces.

It has been assumed that the topological properties of two-dimensional manifolds, from the point of view of Čech cohomology especially, are familiar to the reader. A few words should perhape be added here, in case that is not so. Most books on Riemann surfaces begin with a discussion of the topology of surfaces, usually simplicial or singular homology theory with particular emphasis on the two-dimensional case; and the reader without this background can quite well consult one of these books. (See for example Lars V. Ahlfors and Leo Sario, Riemann Surfaces (Princeton University Press, 1960); George Springer, Introduction to Riemann Surfaces, (Addison-Wesley, 1957); and of couse Hermann Weyl, The Concept of a Riemann Surface, (English translation, Addison-Wesley, 1964).) The topology of surfaces, also from the point of view of singular homology theory, is covered in H. Seifert and W. Threlfall, Lehrbuch der Topologie (Teubner, 1934; Chelsea, 1947); the fundamental group and covering spaces are also treated in detail there.

The Čech cohomology groups of a compact surface (with coefficients in a constant sheaf, such as $\mathbb{Z}$ or $\mathbb{C}$) are isomorphic to the singular or simplicial cohomology groups, and the cohomology groups can be viewed as dual to the homology groups; so the properties of the Čech cohomology groups needed in these lectures can readily be derived from the discussion of the homology of surfaces in the books mentioned above. (A more general discussion can also be found in Samuel Eilenberg and Norman Steenrod, Foundations of Algebraic Topology, (Princeton Univ. Press, 1951).) More directly, the surface can be triangulated; and taking open neighborhoods of the closed two-simplices as an open covering of the surface, most of the results needed follow from a straightforward calculation. Similarly, it follows quite easily that the fundamental group as defined in §9 (b) is isomorphic to the fundamental group as more customarily defined, (as for instance in Seifert and Threlfall). Referring to the discussion on page 186, note that for an Abelian group G it follows that $\mathrm{Hom}(\pi_1(M),G)/G = \mathrm{Hom}(\pi_1(M),G)$, and hence by Lemma 27, $H^1(M,G) \cong \mathrm{Hom}(\pi_1(M),G)$. Indeed, since G is Abelian, $\mathrm{Hom}(\pi_1(M),G) \cong \mathrm{Hom}(H_1(M),G)$, where $H_1(M)$ is the group $\pi_1(M)$ made Abelian, that is, is the quotient of $\pi_1(M)$ by its commutator subgroup; but $H_1(M)$ is the first homology group of the surface, so this is just the familiar duality between homology and cohomology. In particular, $H^1(M,\mathbb{C}) \cong \mathrm{Hom}(H_1(M),\mathbb{C}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{C})$. With this observation, the isomorphism $\delta\colon \Gamma(M,\mathcal{O}^{1,0}) \longrightarrow H^1(M,\mathbb{C})$ discussed in §8 (a) can be put into a more traditional form as follows. Let $(U_0,U_1,\ldots,U_n,U_0)$ be a closed chain at p, in terms of an open covering $\mathcal{U}$ of the surface M; and let λ be a closed path in the set $U_0 \cup U_1 \cup \ldots \cup U_n$ from the point p, representing the same element of $\pi_1(M,p)$ under the obvious isomorphism. Thus selecting points $p_i \in U_i$, the arc λ can be taken as a union of smooth segments λ_i, where λ_i is an arc in $U_i \cup U_{i+1}$ from p_i to p_{i+1}. If $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$ is any Abelian differential form, select functions $f_i \in \Gamma(U_i, \mathcal{O})$ so that $df_i = \varphi$ in U_i. In $U_i \cap U_{i+1}$ note that $d(f_{i+1} - f_i) = 0$, hence $f_{i+1} - f_i = c_{i,i+1}$ for some constant $c_{i,i+1}$; so upon suitably modifying the functions f_i, there is no loss of generality in supposing that $f_i = f_{i+1}$ in $U_i \cap U_{i+1}$ for $i = 0,1,\ldots,n-1$. The resulting function is of course an indefinite integral of φ. Now the element of $\mathrm{Hom}(\pi_1(M),\mathbf{C})$ corresponding to $\delta\varphi \in H^1(M,\mathbf{C})$ assigns to this loop λ the constant $\sum_{i=0}^{n} c_{i,i+1}$; but this merely amounts to the constant $c_{n,0} = f_n - f_0$, which is evidently the value $\int_\lambda \varphi$. Therefore the isomorphism $\Gamma(M, \mathcal{O}^{1,0}) \longrightarrow \mathrm{Hom}(\pi_1(M),\mathbf{C})$, derived from the composition of the isomorphisms $\Gamma(M, \mathcal{O}^{1,0}) \longrightarrow H^1(M,\mathbf{C})$ and $H^1(M,\mathbf{C}) \longrightarrow \mathrm{Hom}(\pi_1(M),\mathbf{C})$, is that which assigns to an Abelian differential form $\varphi \in \Gamma(M, \mathcal{O}^{1,0})$ and a loop $\lambda \in \pi_1(M,p)$ the period $\int_\lambda \varphi$. This justifies the period matrix terminology considered on page 145, and the discussion in the proof of Corollary 3 of Theorem 18. Of course, $\Gamma(M, \mathcal{O}^{1,0})$ can be replaced by the full space of closed differential forms on M, in the analogous discussion of deRham's Theorem in terms of the periods of differential forms.

Finally, a few words should be said about the intersection matrix on a surface, in connection with the discussion in §8 (c). In the usual approach, the explicit form of the intersection matrix X (recall page 154) is derived in the proof of Abel's theorem, or in the related discussion. (See Ahlfors and Sario, pages 319 ff., for instance.) The argument is essentially the following. Suppose the surface M is represented by a polygon with pairs of edges identified, in the normal form, (as in Seifert and Threlfall, pages 135 ff.):



The elements a_i, b_i generate $H_1(M)$; and dual generators can be selected for the cohomology group $H^1(M, \mathbb{C})$. Upon representing these cohomology classes by closed differential forms $\alpha_i, \beta_i \in \Gamma(M, \mathcal{E}^1)$, this condition is that

$$(*) \qquad \int_{a_i} \alpha_j = \delta_i^j \; ; \quad \int_{b_i} \alpha_j = \int_{a_i} \beta_j = 0 \; ; \quad \int_{b_i} \beta_j = \delta_i^j \, ,$$

where δ_i^j is the Kronecker symbol. In terms of this basis, the intersection matrix χ has the entries

$$\chi_{ij} = \iint \alpha_i \wedge \alpha_j \; , \quad \chi_{i,j+g} = \iint \alpha_i \wedge \beta_j \; , \quad \chi_{i+g,j+g} = \iint \beta_i \wedge \beta_j$$

for $i, j = 1, \ldots, g$, where g = genus of M. Upon applying Stokes' theorem a few times, it follows readily from the equations (*) that χ has the desired form. The details can be left to the reader, (compare Ahlfors and Sario, pages 319 ff.).

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